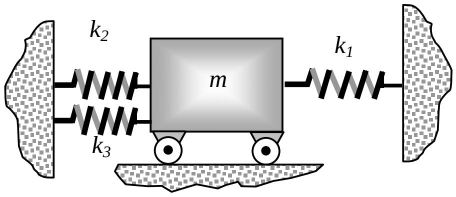
CVEN 363

E6: Equations of Motion - Template

Barroso

Problem 1: Find the Equation of Motion from the *static equilibrium position* for the following systems:



1) Assumptions
- Wheels roll without friction
- Motion will not cause block to hit the supports

2) Define your degree of Freedom
$u(t)$ = horizontal displacement of center of mass of the block from static equilibrium.

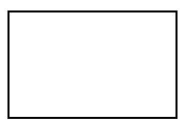
3) Kinematic Constraints – None

We are not told if the springs are pre-stressed or not. In other words, did we have to stretch or compress the springs to fit the system within the walls. So let's check a couple of cases:

Case 1: Assume all springs are initially in tension

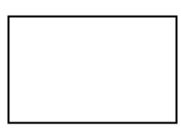
4) Free-Body Diagrams: draw dynamic forces as the system moves from your zero position in the DEFINED positive direction – whether or not you think the system will actually move that way! Don't try and predict the direction of motion at this point. Notice that these systems don't have external forces or given initial conditions, so trying to guess how it will move is pointless.

(a) Static Equilibrium – Forces Acting on System when it is in static equilibrium



where: $\bar{f}_{si}$ = static force on the ith spring $= k_i \Delta_{si}$
k_i = stiffness coefficient of the ith spring
Δ_{si} = deformation of the ith spring (elongation or compression) under static equilibrium

(b) Total-Forces Acting on System
As the block starts to move from the static equilibrium position, the forces in the springs will change due to dynamic motion.



where: f_i = the ***change*** in the spring force due to dynamic behavior =
f_i^T = the ***total*** spring force due to dynamic behavior

(c) Incremental Dynamic Forces (IDF): only those forces that impact dynamic behavior

Note that if you add the forces from the FBD of the static only forces and those from the Incremental Dynamic Forces FBD, you get exactly the FBD showing total forces acting on the system.

Typical question: will you always have to draw all 3 FBD's: NO. You will typically only need one type of FBD: either (b) – total forces, or (c) dynamic only forces. You ARE required to label what type of FBD you are drawing.

5) Apply equilibrium principles

(a) Using FBD showing Total Forces

$$f_I + f_1^T + f_2^T + f_3^T = 0$$
$$f_I + \left(f_1 - \overline{f}_{s1}\right) + \left(f_2 + \overline{f}_{s2}\right) + \left(f_3 + \overline{f}_{s3}\right) = 0$$

Grouping dynamic spring force components together and the static spring force components together:

$$f_I + \left(f_1 + f_2 + f_3\right) + \left(-\overline{f}_{s1} + \overline{f}_{s2} + \overline{f}_{s3}\right) = 0$$

But looking at the Statics FBD, we know that the static components must balance one another. In other words:

$$-\overline{f}_{s1} + \overline{f}_{s2} + \overline{f}_{s3} = 0$$

Substituting that information:

$$f_I + \left(f_1 + f_2 + f_3\right) = 0$$

Notice that this is the EXACT equation we would get if we'd used the Incremental Dynamic Force (IDF) diagram to start with!

Now express the above in term of the degree of freedom and its derivative:

Group terms such that EoM is in standard form: $m^*\ddot{u} + k^*u = p(t)$, where $m^* =$ effective mass and $k^* =$ effective stiffness for the system. So:

Case 2: Assume all springs are initially in compression

4) Free-Body Diagrams: draw dynamic forces as the system moves from your zero position in the DEFINED positive direction – whether or not you think the system will actually move that way! Don't try and predict the direction of motion at this point. Notice that these systems don't have external forces or given initial conditions, so trying to guess how it will move is pointless.

(a) Static Equilibrium – Forces Acting on System when it is in static equilibrium

where: $\overline{f}_{si}$ = static force on the i^th^ spring $= k_i \Delta_{si}$
k_i = stiffness coefficient of the i^th^ spring
Δ_{si} = deformation of the i^th^ spring (elongation or compression) under static equilibrium

(b) Total-Forces Acting on System
As the block starts to move from the static equilibrium position, the forces in the springs will change due to dynamic motion.

where: f_i = the ***change*** in the spring force due to dynamic behavior =
f_i^T = the ***total*** spring force due to dynamic behavior

(c) Incremental Dynamic Forces (IDF): only those forces that impact dynamic behavior

Again, note that if you add the forces from the FBD of the static only forces and those from the Incremental Dynamic Forces FBD, you get exactly the FBD showing total forces acting on the system.

5) Apply equilibrium principles

(a) Using FBD showing Total Forces

$$f_I + f_1^T + f_2^T + f_3^T = 0$$

$$f_I + \left(f_1 + \overline{f}_{s1} \right) + \left(f_2 - \overline{f}_{s2} \right) + \left(f_3 - \overline{f}_{s3} \right) = 0$$

Grouping dynamic spring force components together and the static spring force components together:

$$f_I + \left(f_1 + f_2 + f_3 \right) + \left(+\overline{f}_{s1} - \overline{f}_{s2} - \overline{f}_{s3} \right) = 0$$

But looking at the Statics FBD, we know that the static components must balance one another. In other words:

$$-\overline{f}_{s1} + \overline{f}_{s2} + \overline{f}_{s3} = 0$$

Substituting that information, we now get:

$$f_I + \left(f_1 + f_2 + f_3 \right) = 0$$

Notice that this is the EXACT equation we would get if we'd used the Incremental Dynamic Force (IDF) diagram to start with!

Now express the above in term of the degree of freedom and its derivative:

Group terms such that EoM is in standard form: $m^* \ddot{u} + k^* u = p(t)$, where $m^* =$ effective mass and $k^* =$ effective stiffness for the system. So:

Comparing Case 1 and Case 2: We get the exact same EoM!

Notice that

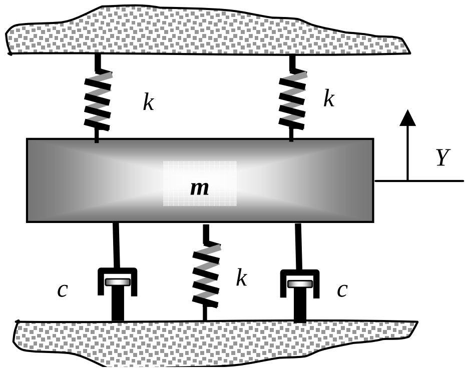
- We got the exact same EoM independent of the assumption we made regarding the initial pre-stressing in the springs.
- The static component of the spring forces did not show up in our EoM.

This leads us to a couple of key points:

- As long as *we can **see ahead** that the static forces will cancel*, we can draw only the FBD showing the incremental dynamic forces (forces impacting dynamics) & get the correct EoM. This does NOT mean that we are neglecting the static forces. If you are in doubt – use TOTAL forces.

- In general, if we define our degree of freedom as being measured from the static equilibrium position then the static forces (and gravity – we'll see this later) will not impact the dynamics and so not be part of the EoM.

Problem 1b: Find the EoM of the following system.



1) Assumptions: Assume no deformation in the springs to make system fit within the supports (no static pre-stressing)

<u>Possible Reference Positions for our Degree of Freedom</u>

We can measure degree of freedom relative to ANY point in the plane. There are two options that are by far the most common:

- Motion from point in plane where all the springs are undeformed
- Motion from point in plane where the mass would be if in static equilibrium

So this leads us to two possible degree of freedom definitions:

$Y =$

$z =$

Relationship between coordinates:

Depending on your choice of reference position – i.e. on your choice of dof – you will get a slightly different EoM. So being clear on your choice is **CRITICAL**.

<u>Case 1: Choose Undeformed Spring Position as Reference – Y</u>

2) Define your degree of Freedom

3) Kinematic Constraints – None

4) Free-Body Diagram

(a) Total-Forces Acting on System – only real option when using undeformed position

5) Apply equilibrium principles

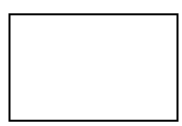
<u>Case 2: Choose Equilibrium Spring Position as Reference – z</u>

2) Define your degree of Freedom

3) Kinematic Constraints – None

4) Free-Body Diagrams

(a) Static Equilibrium – Forces Acting on System when it is in static equilibrium



where: $\overline{f}_{si}$ = static force on the ith spring $= k_i \Delta_{si}$

k_i = stiffness coefficient of the ith spring

Δ_{si} = deformation of the ith spring (elongation or compression) under static equilibrium

(b) Total-Forces Acting on System

where: f_i = the ***change*** in the spring force due to dynamic behavior =

f_i^T = the ***total*** spring force due to dynamic behavior

(c) Incremental Dynamic Forces (IDF): only those forces that impact dynamic behavior

5) Apply equilibrium principles

Comparing Case 1 and Case 2: We DO NOT get the exact same EoM!

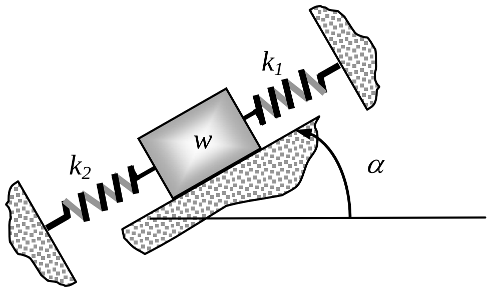
Notice that

- The left-hand side (LHS) of the EoM is exactly the same. The differences are on the RHS.

This leads us to a couple of key points:

- The EoM you get will be dependent on your choice of DoF.
- When using dof from static equilibrium, most times gravity will cancel out with the static spring forces and not be a part of your EoM
    - Exception: when gravity is a part of the restoring or the destabilizing force (pendulums)
- Drawing a FBD showing total forces works for both DoF choices. However, the incremental dynamic forces (IDF) diagram is only appropriate when using a dof from static equilibrium.
- The two EoMs are completely equivalent. Plug-in the coordinate relationship we defined in the beginning into the EoM from the undeformed position. You will get the EoM from static equilibrium position.

Problem 1c: Find the Equation of Motion from the ***static equilibrium position*** for the following system:



1) Assumptions
   - Block slides on frictionless surface
   - Block will not collide with supports during motion

2) Define your degree of Freedom
$u(t) =$ motion of block along incline from static equilibrium position

3) Kinematic Constraints – None

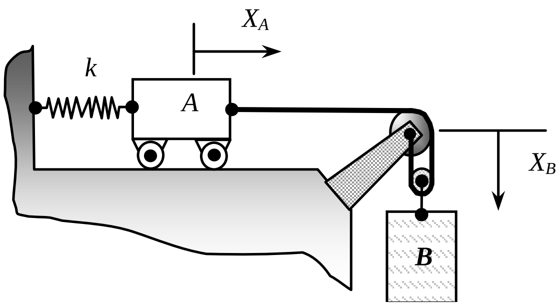
4) Free-Body Diagram(s)

5) Apply equilibrium principles

Resulting EoM:

$$m\ddot{u} + (k_1 + k_2)u = 0$$

Problem 1d: Find the Equation of Motion from the *static equilibrium position* for the following system:



1) Assumptions

2) Define your degree of Freedom

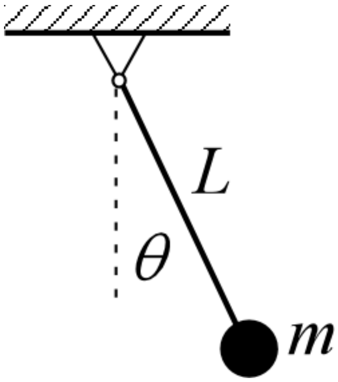
3) Kinematic Constraints

4) Free-Body Diagram

5) Apply equilibrium principles

Resulting EoM:

Problem 2a: Write the exact equation of motion for the following pendulums using the rotation of the bar as your degree of freedom. Then linearize the equation for small angle motions.



1) Assumptions
    - the bar is massless and rigid
    - No friction at pivot

2) Define your degree of Freedom

3) Kinematics –

4) Free-Body Diagram - Total-Forces Acting on System

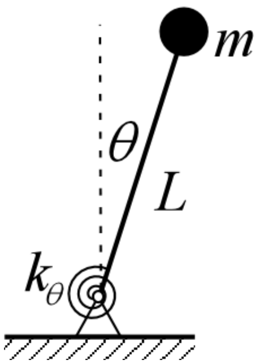
5) Apply equilibrium principles

6) Fully non-linear EoM

7) Linearization assumption: small angles

8) Linearize EoM

Problem 2b: Write the exact equation of motion for the following pendulums using the rotation of the bar as your degree of freedom. Then linearize the equation for small angle motions.



1) Assumptions
- the bar is massless and rigid
- rotational spring undeformed when vertical

2) Define your degree of Freedom

3) Kinematics

4) Free-Body Diagram – Total-Forces Acting on System

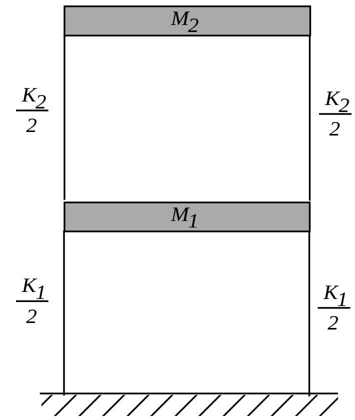
5) Apply equilibrium principles

6) Fully non-linear EoM

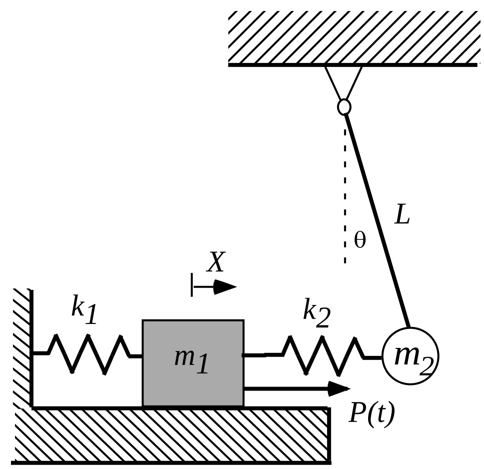
7) Linearization assumption: small angles

8) Linearized EoM

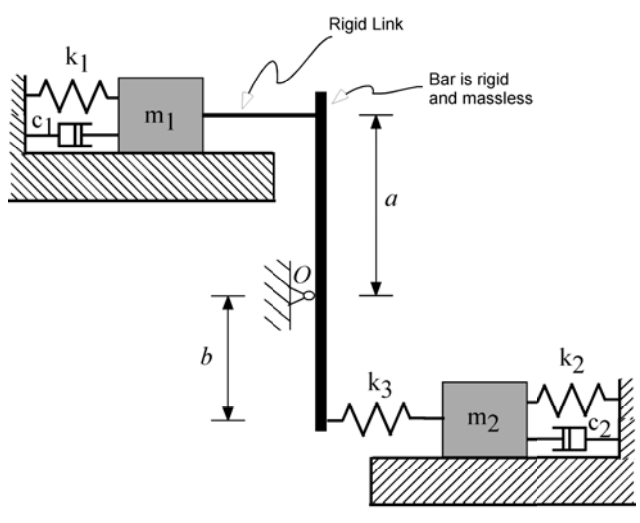
Problem 1: Formulate the differential equation of motion for the following system. Write your answer in matrix format.

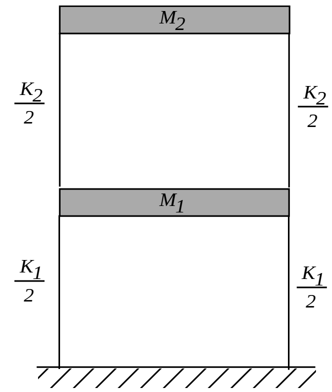


Problem 2: (old exam problem) Find the Equations of Motion for the following system. **Solve this problem using both equilibrium and energy (Lagrange's) methods.** Place the answer in matrix format. You may assume that: springs are undeformed when pendulum is vertical, all springs and dampers retain their horizontal alignment, and that the rotation about pivot point is small.



Problem 3: (old exam problem) Find the Equations of Motion for the following system for small rotations of the rigid bar. Place the answer in matrix format. You may assume that all springs and dampers retain their horizontal alignment and that the rotation about pivot point, *O*, is small.





Problem 1: Find the Equation of Motion for the following systems:

1) Assumptions
   a. Total Mass of the structure is concentrated at the floor levels (lumped mass model)
   b. The beams and floor system assumed to be infinitely rigid compared to the columns, so all story stiffness comes from lateral deformation of the columns
   c. Neglects the effects of axial forces on the columns

   Resulting simplified physical model:

2) Define your degree(s) of Freedom
- How many? You need as many EoMs as you have degrees of freedom.
- What are they? Direction, reference position, total or relative?

3) Kinematic Constraints – None

4) Free-Body Diagrams

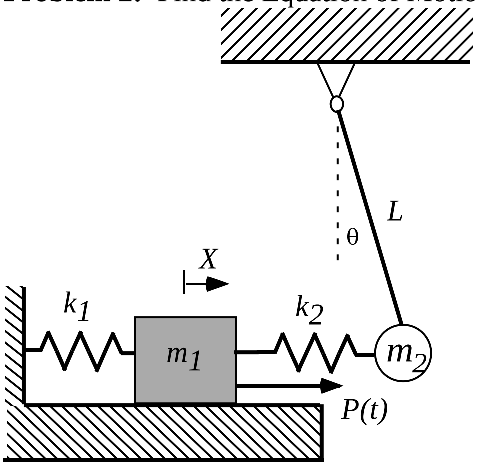
5) <u>Apply Equilibrium</u>

5) <u>Express in Matrix Format (if linear system)</u>: $\left[M\right]\left\{\ddot{x}\right\}+\left[C\right]\left\{\dot{x}\right\}+\left[K\right]\left\{x\right\}=\left\{F\left(t\right)\right\}$

Must express it so that:

- Mass matrix is symmetric with positive values on the diagonal. Mass matrix is a positive definite matrix.
- Stiffness matrix is symmetric. Diagonal elements:
  - All Positive: system is stable
  - Negative: system unstable or only conditionally stable
- Damping matrix is symmetric with positive values on the diagonal (for our systems).

Problem 2: Find the Equation of Motion for the following systems:



1) Assumptions

2) Define your degree(s) of Freedom

- How many? You need as many EoMs as you have degrees of freedom.
- What are they? Direction, reference position, total or relative?

3) Kinematic Constraints

APPROACH I: EQUILIBRIUM BASED METHOD

4) Free-Body Diagrams

5) <u>Apply Equilibrium</u>

5) <u>Express in Matrix Format (if linear system)</u>: $\left[M\right]\left\{\ddot{x}\right\}+\left[C\right]\left\{\dot{x}\right\}+\left[K\right]\left\{x\right\}=\left\{F\left(t\right)\right\}$

Must express it so that:

- Mass matrix is symmetric with positive values on the diagonal. Mass matrix is a positive definite matrix.
- Stiffness matrix is symmetric. Diagonal elements:
    - All Positive: system is stable
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APPROACH II: ENERGY BASED METHOD – LAGRANGE'S

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_i}T\right)-\frac{\partial}{\partial q_i}T+\frac{\partial}{\partial q_i}V-Q=0$$

4) <u>Energy Equations</u>

- Kinetic Energy, T

- Potential Energy, V

5) <u>Lagrange's Equation for 1st Degree of Freedom:</u> $q_1 = x$

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}_i}T\right)-\frac{\partial}{\partial x_i}T+\frac{\partial}{\partial x_i}V-Q=0$$

- Partial Derivatives

$\frac{\partial}{\partial x}T$

$\frac{\partial}{\partial \dot{x}}T$

$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}T\right)$

$\frac{\partial}{\partial x}V$

- Non-conservative forces:

- Substituting into Lagrange's:

6) Lagrange's Equation for 2nd Degree of Freedom: $q_2 = \theta$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}T\right)-\frac{\partial}{\partial\theta}T+\frac{\partial}{\partial\theta}V-Q=0$$

- Partial Derivatives

$$\frac{\partial}{\partial\theta}T$$

$$\frac{\partial}{\partial\dot{\theta}}T$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\theta}}T\right)$$

$$\frac{\partial}{\partial\theta}V$$

- Non-conservative forces:

- Substituting into Lagrange's:

7) Express in Matrix Format (if linear system): $[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F(t)\}$

Do we get the same EoM as using Equlibrium?