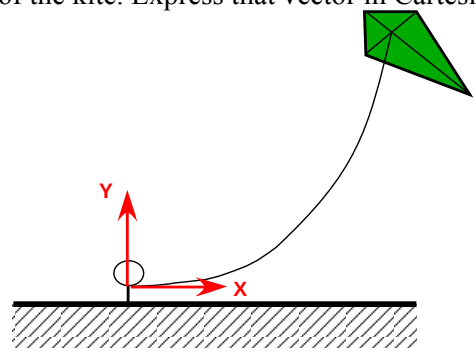
Problem 1: A kite is let out by a pulley system on the ground. The equation of the kite's path can at any time be described by: $y = (x^2)/25$ ft. The vertical position is defined as $y = 15t$. At $t = 1.5$ seconds, find:

a) The position vector for the kite in Cartesian and Polar coordinates. Take the origin at the pulley
b) The straight distance between the kite and the pulley
c) The velocity vector of the kite. Express that vector in Cartesian, Polar, and Path coordinates
d) The acceleration vector of the kite. Express that vector in Cartesian, Polar, and Path coordinates



Given:

$y = \dfrac{x^2}{25}$ ft → Path equation. Information most closely related to Path coordinates

$y = 15t$ ft → Y as a function of time. Most closely related to Cartesian coordinates

Part (a)

Required: The position vector {r} for the kite in Cartesian and Polar coordinates.

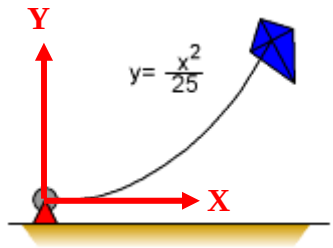
Solution:
Need to determine in what coordinate system the problem should be solve in first. Part of given information is in Path coordinates and part is in Cartesian coordinates.

To solve in Path: need to find speed and acceleration tangent to path before proceeding. Arguably the harder coordinate system than Cartesian.

To solve in Cartesian: need to find x as a function of time. If we have both x and y as functions of time, then easy to solve in Cartesian. To express x as function of time:

$$y = \frac{x^2}{25}$$
$$(15t) = \frac{x^2}{25}$$
$$x^2 = 25(15t) = 375t$$
$$x = \sqrt{375t}$$

Choice: solve problem in Cartesian coordinates first! Draw coordinate system on the figure to indicate location of origin and positive directions for x and y.



<u>Position in Cartesian:</u> By definition:

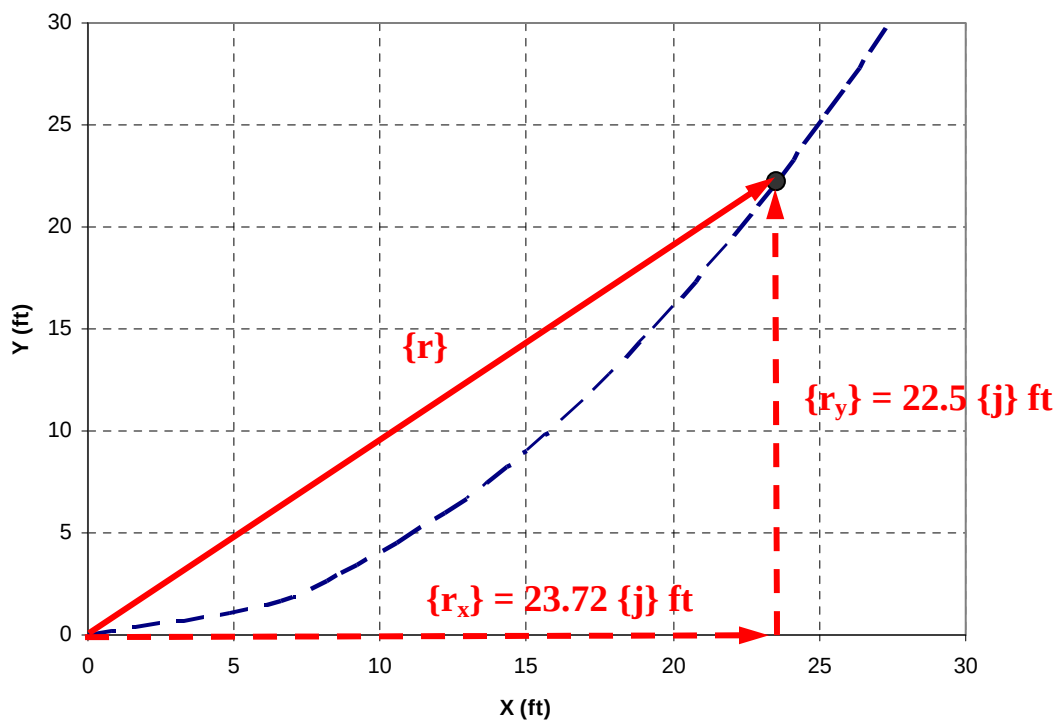
$$\{r\} = x(t)\{i\} + y(t)\{j\}$$
$$= \left(\sqrt{375t}\right)\{i\} + (15t)\{j\} \quad \text{ft}$$

At t = 1.5 seconds:

$$\{r\} = \left(\sqrt{375*1.5}\right)\{i\} + (15*1.5)\{j\} \quad \text{ft}$$
$$\{r\} = 23.72\{i\} + 22.5\{j\} \quad \text{ft}$$

Note that the answer ONLY makes sense with the coordinate system shown above!!

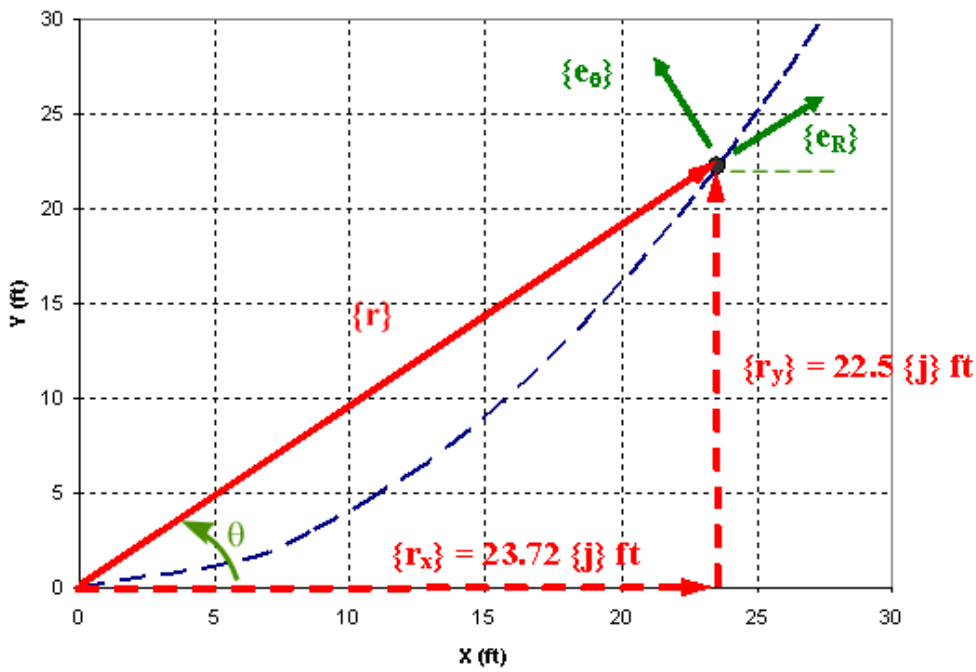
Answer: $\boxed{\{r\} = 23.72\{i\} + 22.5\{j\} \quad \text{ft}}$ Showing the vector graphically:



<u>Position in Polar:</u> We want to express the SAME position vector {r} but in a different coordinate system. So locate Polar coordinates:

- Positive Radial direction goes from origin outward towards the point
- Positive Theta direction is 90° counter-clockwise from Radial

So first we need to determine those coordinate directions. Radial will be exactly in the same line as the position vector {r} we already found. I'll draw the unit vectors at the point (as that is easier to visualize), but the origin of the Polar Coordinates is still at the pulley.



To determine the angle between the positive x-direction and the positive radial direction, we can look at the right triangle formed by the components of {r}:

$$\tan\theta = \frac{|r_y|}{|r_x|} = \frac{22.5}{23.72}$$

$$\theta = \tan^{-1}(0.9486)$$

$$\theta = 43.49^\circ$$

In Polar coordinates, the entire position vector lies along the Radial direction (that is how the coordinate system is defined). Now that we have located that direction with the unit vectors in the figure above, we can write the Position Vector as:

$$\{r\} = \sqrt{(23.72)^2 + (22.5)^2}\,\{e_R\} \quad \text{ft}$$

$$\boxed{\{r\} = 32.69\{e_R\} \quad \text{ft}}$$

Key point: in both coordinate system cases we are representing the exact same vector {r}

Part (b)

Required: The straight distance between the kite and the pulley.

Solution: The distance is the length of the position vector. Can be found from the expression in Cartesian or in Polar.

<u>Option 1: From {r} written in Cartesian</u>

$$\text{distance} = \text{magnitude of } \{\mathrm{r}\} = \|\{r\}\|$$
$$\|\{r\}\| = \sqrt{(23.72)^2 + (22.5)^2} \quad \text{ft}$$
$$\|\{r\}\| = 32.69 \quad \text{ft}$$

<u>Option 2: From {r} written in Polar</u>

$$\text{distance} = \text{magnitude of } \{\mathrm{r}\} = \|\{r\}\|$$
$$\|\{r\}\| = \sqrt{(32.69)^2} \quad \text{ft}$$
$$\|\{r\}\| = 32.69 \quad \text{ft}$$

Key point: in both coordinate system cases we are representing the exact same vector {r}, so their magnitudes/lengths MUST match. Should be used as a check to make sure your answers are correct when expressing any vector in multiple coordinates.

Part (c)

Required: The velocity vector of the kite. Express that vector in Cartesian, Polar, and Path coordinates.

Solution: Choose a coordinate system to start. As with position, Cartesian will be easiest since have x and y as functions of time.

We know position vector is:

$$\{r\} = x(t)\{i\} + y(t)\{j\} = \left(\sqrt{375t}\right)\{i\} + (15t)\{j\} \quad \text{ft}$$

<u>Velocity vector in Cartesian:</u> By definition:

$$\{v\} = \dot{x}(t)\{i\} + \dot{y}(t)\{j\}$$

Need to take derivatives of x and y as functions of time.

$$\dot{x} = \frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{375t}\right) = \frac{d}{dt}\left((375t)^{1/2}\right)$$
$$\dot{x} = \frac{1}{2}(375t)^{-1/2}(375)$$
$$\dot{x} = \frac{187.5}{\sqrt{375t}} \ \text{ft}/\text{s}$$

$$\dot{y} = \frac{dy}{dt} = \frac{d}{dt}(15t)$$
$$\dot{y} = 15 \ \text{ft}/\text{s}$$

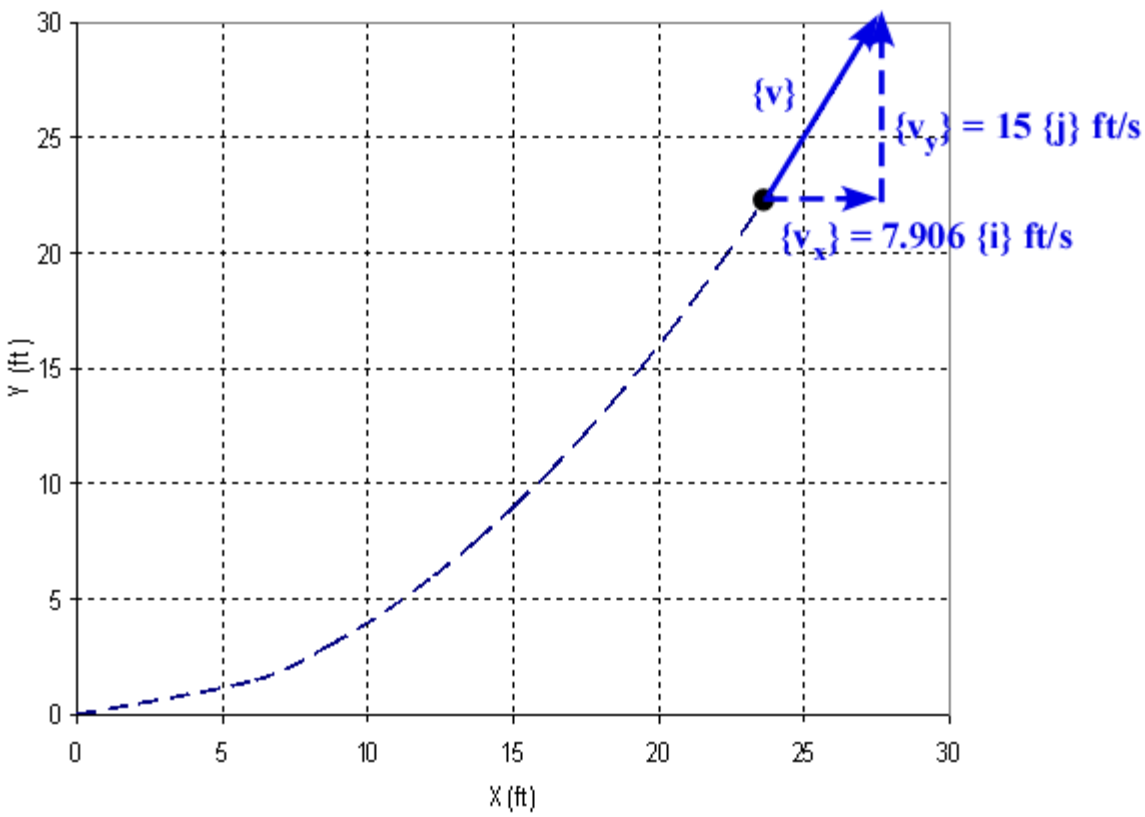
Substituting these expressions into the Velocity vector in Cartesian is:

$$\{v\} = \frac{187.5}{\sqrt{375t}}\{i\} + 15\{j\} \ \text{ft}/\text{s}$$

At t = 1.5 seconds:

$$\{v\} = \frac{187.5}{\sqrt{375*1.5}}\{i\} + 15\{j\} \ \text{ft}/\text{s}$$
$$\boxed{\{v\} = 7.906\{i\} + 15\{j\} \ \text{ft}/\text{s}}$$

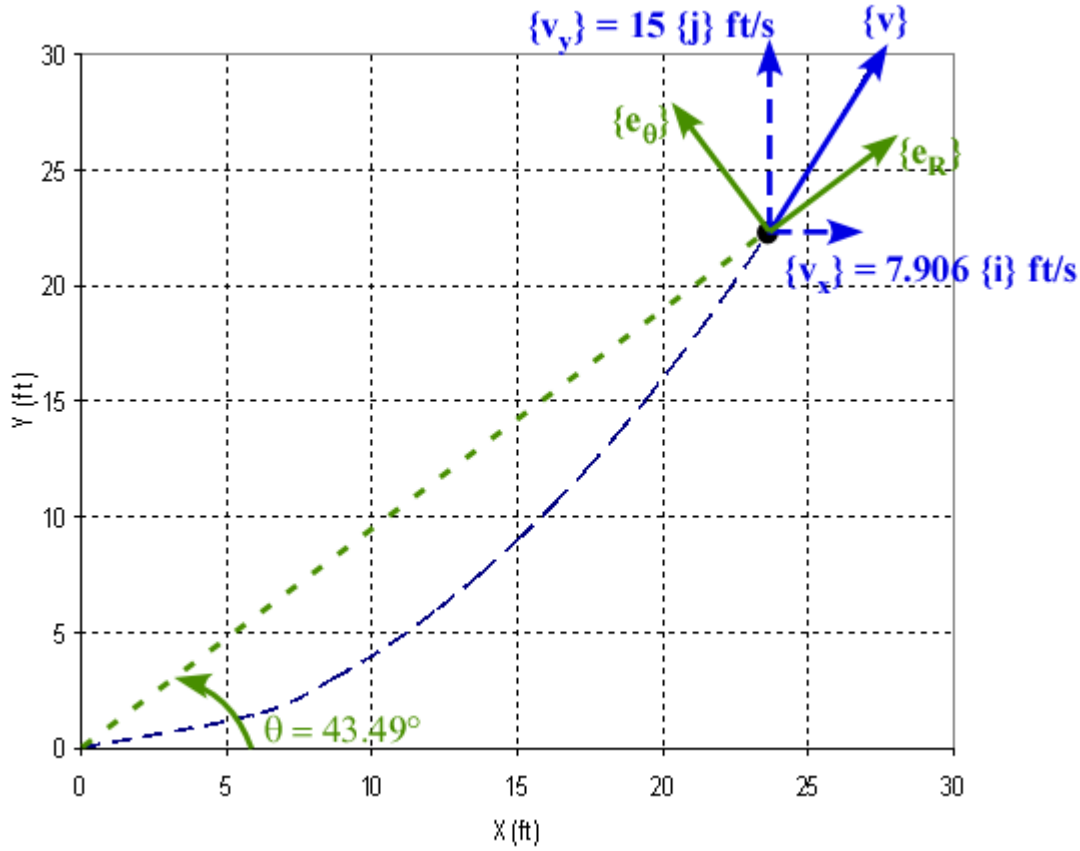
Graphically:



Though not asked, lets determine the speed (magnitude of {v}) as that will be the same in all 3 coordinate systems. This will let us know if we make a mistake along the way.

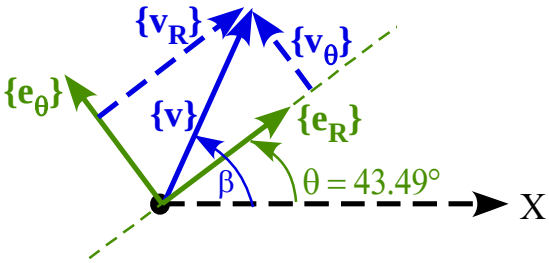
$$\text{speed} = \|\{v\}\| = \sqrt{(v_x)^2 + (v_y)^2} = \sqrt{(7.906)^2 + (15)^2} \ \text{ft}/\text{s}$$
$$\|\{v\}\| = 16.96 \ \text{ft}/\text{s}$$

Velocity vector in Polar: We want to express the SAME velocity vector {v} but in a different coordinate system. So need locate Polar coordinates – which we have done in Part (a). Looking at the velocity vector together with the unit vectors for Polar:



In order to express the vector {v} in Polar, we need to find the component directions of {v} along $\{e_R\}$ and $\{e_\theta\}$. We can do this 2 ways:

1. Tackling the full vector {v}:



Determine angle between {v} and x-dir:

$$\tan\beta = \frac{|v_y|}{|v_x|} = \frac{15}{7.906}$$

$$\beta = \tan^{-1}(1.897)$$

$$\beta = 62.2^\circ$$

To find components, use the difference between the angles θ and β:

$\{v_R\} = \|\{v\}\|\cos(\beta-\theta)\{e_R\}$ Note: these equation depends on the geometry of the problem.

$\{v_\theta\} = \|\{v\}\|\sin(\beta-\theta)\{e_\theta\}$ Draw the figure!!

Substituting values:

$$\{v_R\} = (16.96)\cos\left(62.2^\circ - 43.49^\circ\right)\{e_R\}\ \text{ft}/\text{s}$$
$$\{v_R\} = (16.96)\cos\left(18.71^\circ\right)\{e_R\}\ \text{ft}/\text{s}$$
$$\{v_R\} = 16.1\{e_R\}\ \text{ft}/\text{s}$$

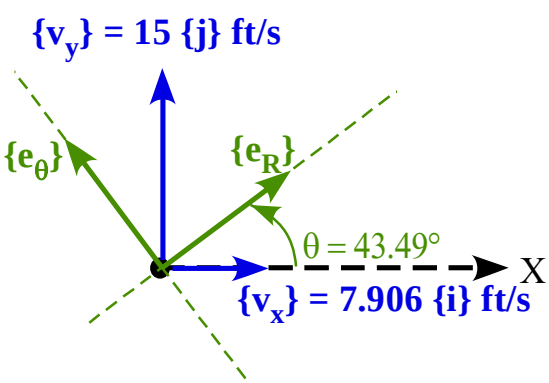
$$\{v_\theta\} = (16.96)\sin\left(62.2^\circ - 43.49^\circ\right)\{e_\theta\}\ \text{ft}/\text{s}$$
$$\{v_\theta\} = (16.96)\sin\left(18.71^\circ\right)\{e_\theta\}\ \text{ft}/\text{s}$$
$$\{v_\theta\} = 5.44\{e_\theta\}\ \text{ft}/\text{s}$$

So full vector in Polar:

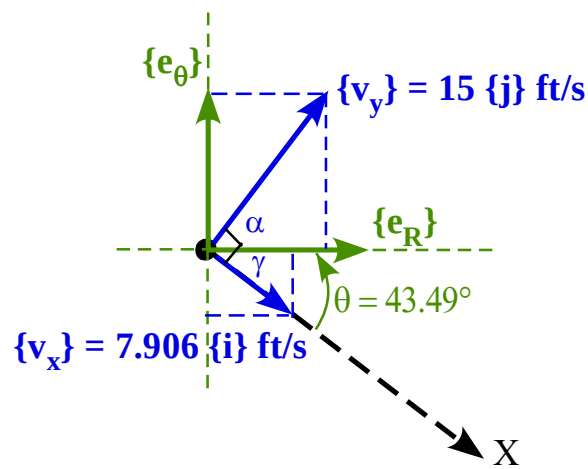
$$\{v\} = \{v_R\} + \{v_\theta\}$$
$$\{v\} = 16.1\{e_R\} + 5.44\{e_\theta\}\ \text{ft}/\text{s}$$

2. <u>Tackling individual vector components $\{v_x\}$ and $\{v_y\}$ and adding their contributions:</u>

Two different ways to draw: (a) oriented with x and y as in original picture, or (b) oriented so your NEW axes lie in horizontal and vertical directions.



Option (a)



Option (b)

Since the x and y directions (and therefore components) are perpendicular to one another, we can determine the angles:

$$\alpha + \gamma = 90^\circ$$
$$\gamma = \left(\theta = 43.49^\circ\right)$$
$$\alpha = 90^\circ - 43.49^\circ = 46.51^\circ$$

Both $\{v_x\}$ and $\{v_y\}$ vectors will contribute (have components) along the R and Θ directions. So:

$$\{v_R\} = \left(\|\{v_x\}\|\cos(\gamma) + \|\{v_y\}\|\cos(\alpha)\right)\{e_R\}$$

$$\{v_\theta\} = \left(-\|\{v_x\}\|\sin(\gamma) + \|\{v_y\}\|\sin(\alpha)\right)\{e_\theta\}$$

<u>Note</u>: these equation depends on the geometry of the problem. Draw the figure!!

Substituting values:

$$\{v_R\} = \left(|7.906|\cos(43.49) + |15|\cos(46.51)\right)\{e_R\}\ \text{ft}/\text{s}$$

$$\{v_R\} = (5.736 + 10.32)\{e_R\}\ \text{ft}/\text{s}$$

$$\{v_R\} = 16.1\{e_R\}\ \text{ft}/\text{s}$$

$$\{v_\theta\} = \left(-|7.906|\sin(43.49) + |15|\sin(46.51)\right)\{e_\theta\}\ \text{ft}/\text{s}$$

$$\{v_\theta\} = (-5.441 + 10.88)\{e_\theta\}\ \text{ft}/\text{s}$$

$$\{v_\theta\} = 5.44\{e_\theta\}\ \text{ft}/\text{s}$$

So full vector in Polar:

$$\{v\} = \{v_R\} + \{v_\theta\}$$

$$\{v\} = 16.1\{e_R\} + 5.44\{e_\theta\}\ \text{ft}/\text{s}$$

Matches exactly the answer with other approach! So

$$\boxed{\{v\} = 16.1\{e_R\} + 5.44\{e_\theta\}\ \text{ft}/\text{s}}$$

Again, answer is only complete because we have clearly defined and drawn the unit vector directions for Polar coordinates previously.

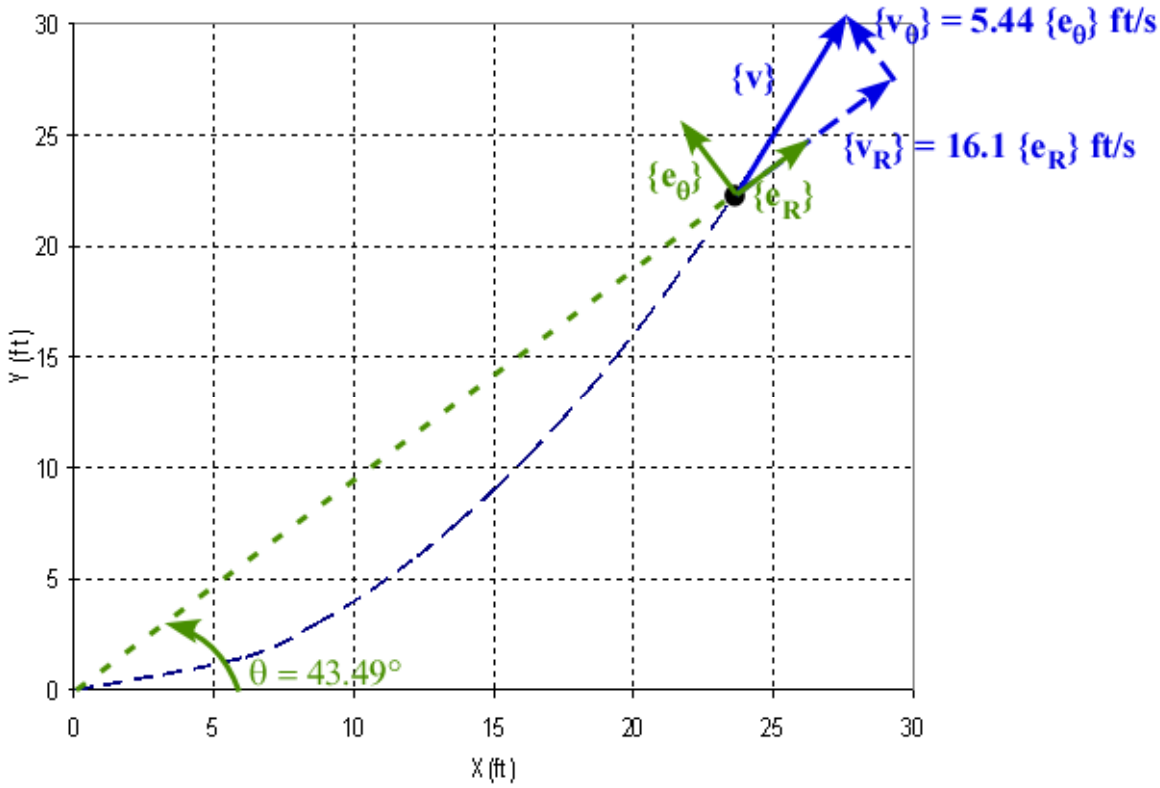
Check – compute the speed:

$$\text{speed} = \|\{v\}\| = \sqrt{(v_R)^2 + (v_\theta)^2} = \sqrt{(16.1)^2 + (5.44)^2} = \text{ft}/\text{s}$$

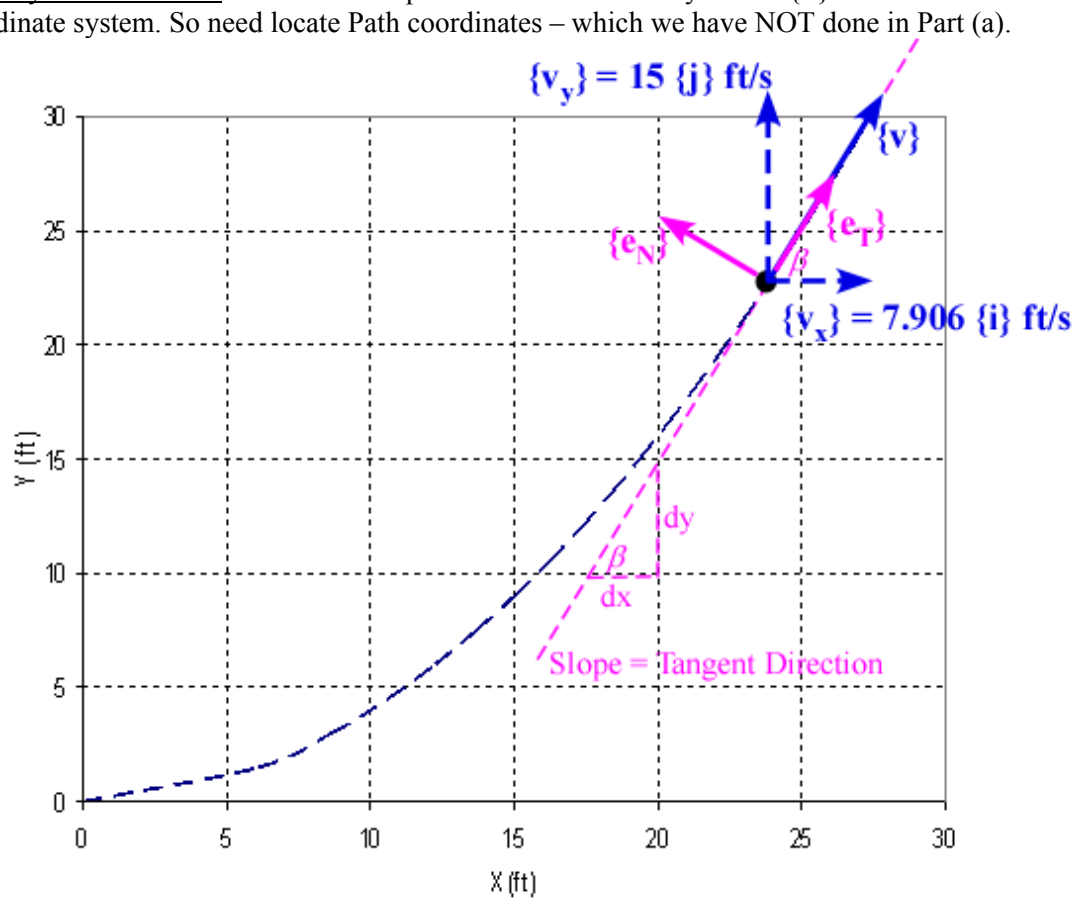
$$\|\{v\}\| = 16.96\ \text{ft}/\text{s}$$

Matches our previous answer.

Graphically, we can see the components:



<u>Velocity vector in Path:</u> We want to express the SAME velocity vector {v} but in a different coordinate system. So need locate Path coordinates – which we have NOT done in Part (a).



Two ways to locate Tangential direction in Path:

1. Determining the slope/tangent to the curve. Requires that we have a path equation (y as a function of x)
2. Utilizing the fact that the velocity vector lies exactly in the tangential direction

We are given the path equation. However, we also DO know the velocity vector. So for this problem, either approach will work to determine PATH coordinates.

Looking at the velocity vector together with the unit vectors for Path:

1. Determining the slope to the curve: Our Path equation is:

$$y = \frac{x^2}{25} \text{ ft}$$

To find slope, take derivative with respect to x:

$$\frac{d}{dx}y = \frac{d}{dx}\left(\frac{x^2}{25}\right)$$

$$y' = \frac{2x}{25} = \frac{x}{12.5}$$

Note: derivative with respect to x is NOT the same as with respect to time. So

$$\frac{dy}{dx} \neq \frac{dy}{dt}$$

$$y' \neq \dot{y}$$

Evaluating the slope at the point when t = 1.5 seconds, we use the value of x at that time.

$$x = \sqrt{375t} = \sqrt{375*1.5}$$

$$x = 23.72 \text{ ft}$$

Substituting into slope equation:

$$y' = \frac{23.72}{12.5} = 1.897 \text{ ft}/\text{ft}$$

Using that information to find β (angle between positive x and tangent direction):

$$\tan\beta = \left(\frac{dy}{dx} = y'\right)$$

$$\tan\beta = 1.897$$

$$\beta = 62.2^\circ$$

2. Velocity vector lies exactly in the tangential direction:

Determine angle between {v} and x-dir:

$$\tan\beta = \frac{|v_y|}{|v_x|} = \frac{15}{7.906}$$

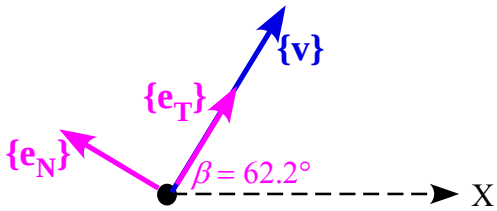
$$\beta = \tan^{-1}(1.897)$$

$$\beta = 62.2^\circ$$

Both approaches give us the same information. For some problems, you may only have one possible approach for which you have enough information.

We now can express the velocity vector in Path. Again, two approaches:

1. <u>Velocity vector lies exactly in the tangential direction:</u> So we have a vector of length equal to the speed and exactly along T direction:

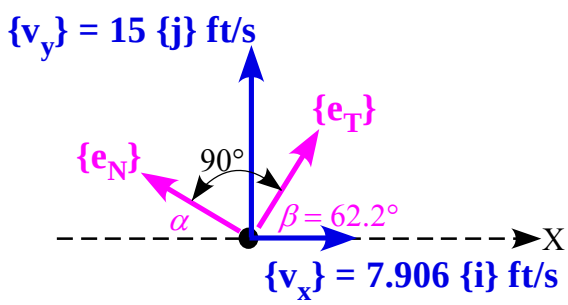


$$\text{speed} = \|\{v\}\| = 16.96\ \text{ft}/\text{s}$$

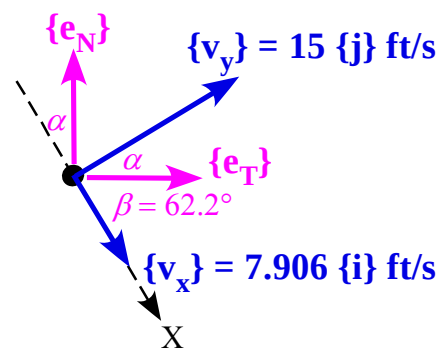
$$\{v\} = 16.96\{e_T\}\ \text{ft}/\text{s}$$

2. <u>Tackling individual vector components $\{v_x\}$ and $\{v_y\}$ and adding their contributions:</u>

Two different ways to draw: (a) oriented with x and y as in original picture, or (b) oriented so your NEW axes lie in horizontal and vertical directions.



Option (a)



Option (b)

Since the x and y directions (and therefore components) are perpendicular to one another, we can determine the angles:

$$\alpha + \beta = 90^\circ \quad \& \quad \beta = 62.2^\circ$$

$$\alpha = 90^\circ - 62.2^\circ$$

$$\alpha = 27.8^\circ$$

Both $\{v_x\}$ and $\{v_y\}$ vectors will contribute (have components) along the T and N directions. So:

$$\{v_T\} = \left(\|\{v_x\}\|\cos(\beta) + \|\{v_y\}\|\cos(\alpha)\right)\{e_T\}$$

$$\{v_\theta\} = \left(-\|\{v_x\}\|\sin(\beta) + \|\{v_y\}\|\sin(\alpha)\right)\{e_N\}$$

<u>Note</u>: these equation depends on the geometry of the problem. Draw the figure!!
Substituting values:

$$\{v_T\} = \left(|7.906|\cos(62.2) + |15|\cos(27.8)\right)\{e_T\}\ \text{ft}/\text{s}$$
$$\{v_T\} = (3.687 + 13.67)\{e_T\}\ \text{ft}/\text{s}$$
$$\{v_T\} = 16.96\{e_T\}\ \text{ft}/\text{s}$$

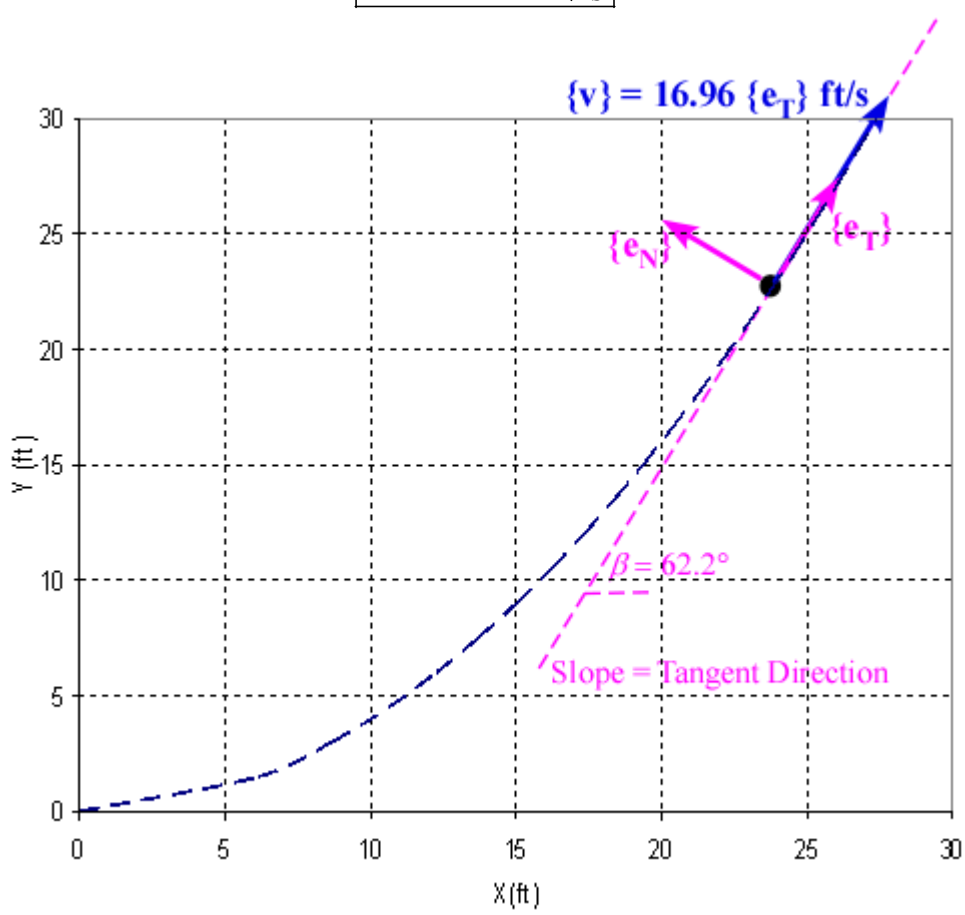
$$\{v_N\} = \left(-|7.906|\sin(62.2) + |15|\sin(27.8)\right)\{e_N\}\ \text{ft}/\text{s}$$
$$\{v_N\} = (-6.99 + 6.99)\{e_N\}\ \text{ft}/\text{s}$$
$$\{v_N\} = 0\{e_N\}\ \text{ft}/\text{s}$$

Combining:

$$\{v\} = \{v_T\} + \{v_N\} = 16.96\{e_T\}\ \text{ft}/\text{s} + 0\{e_N\}\ \text{ft}/\text{s}$$
$$\{v\} = 16.96\{e_T\}\ \text{ft}/\text{s}$$

Both approaches give us the same answer in Path coordinates. So the velocity vector is:

$$\boxed{\{v\} = 16.96\{e_T\}\ \text{ft}/\text{s}}$$



Part (d)

Required: The acceleration vector of the kite. Express that vector in Cartesian, Polar, and Path coordinates

Solution: Choose a coordinate system to start. As with position, Cartesian will be easiest since have x and y as functions of time.

We know velocity vector is:

$$\{v\} = \frac{187.5}{\sqrt{375t}}\{i\} + 15\{j\}\ \text{ft}/\text{s}$$

<u>Acceleration vector in Cartesian:</u> By definition:

$$\{a\} = \ddot{x}(t)\{i\} + \ddot{y}(t)\{j\}$$

Need to take derivatives of $\dot{x}$ and $\dot{y}$ as functions of time.

$$\ddot{x} = \frac{d}{dt}(\dot{x}) = \frac{d}{dt}\left(\frac{187.5}{\sqrt{375t}}\right)\text{ft}/\text{s}^2$$

$$\ddot{x} = \frac{d}{dt}\left(187.5(375t)^{-1/2}\right)\text{ft}/\text{s}^2$$

$$\ddot{x} = 187.5\left(\frac{-1}{2}\right)(375t)^{-3/2}(375)\text{ft}/\text{s}^2$$

$$\ddot{x} = (-35156.2)(375t)^{-3/2}$$

$$\ddot{x} = \frac{-35156.2}{(375t)^{3/2}}$$

$$\ddot{y} = \frac{d}{dt}(\dot{y}) = \frac{d}{dt}(15)\text{ft}/\text{s}^2$$

$$\ddot{y} = 0\ \text{ft}/\text{s}^2$$

Evaluating at t = 1.5sec

$$\ddot{x} = \frac{-35156.2}{(375*1.5)^{3/2}} = \frac{-35156.2}{(562.5)^{3/2}}$$

$$\ddot{x} = \frac{-35156.2}{13340.86}$$

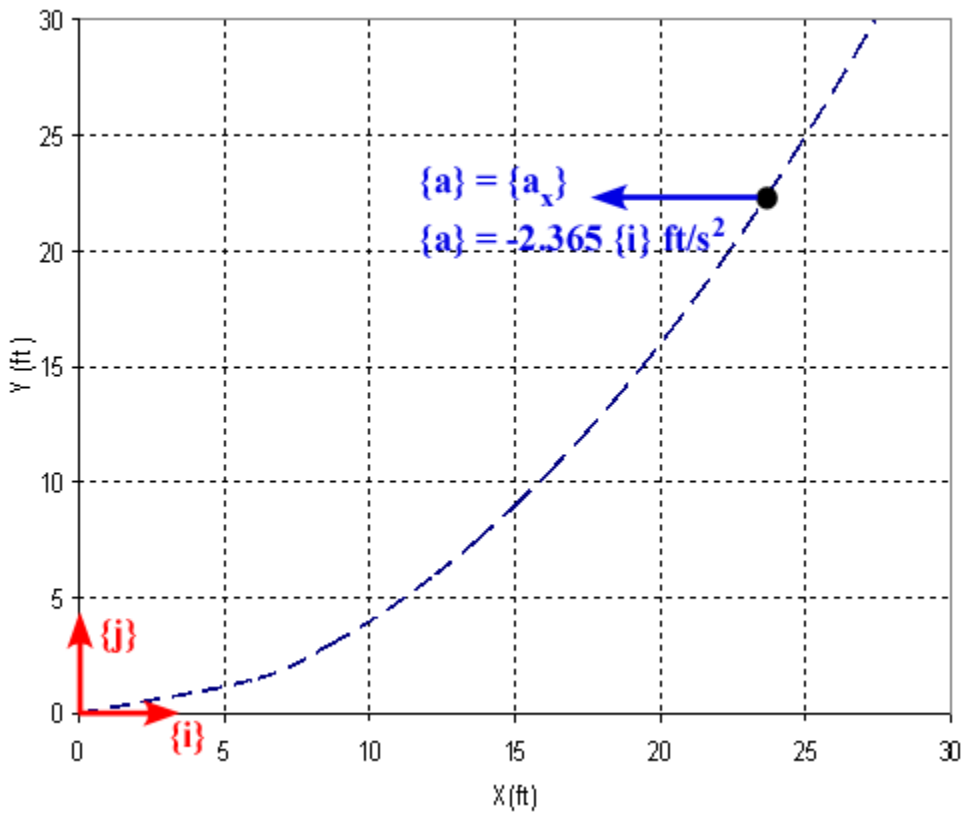
$$\ddot{x} = -2.635\ \text{ft}/\text{s}^2$$

$$\ddot{y} = 0\ \text{ft}/\text{s}^2$$

Substituting values, the acceleration vector at t = 1.5 s is then:

$$\boxed{\{a\} = -2.635\{i\} + 0\{j\}\ \text{ft}/\text{s}^2}$$

Graphically:



Though not asked, lets determine magnitude of {a} as that will be the same in all 3 coordinate systems. This will let us know if we make a mistake along the way.

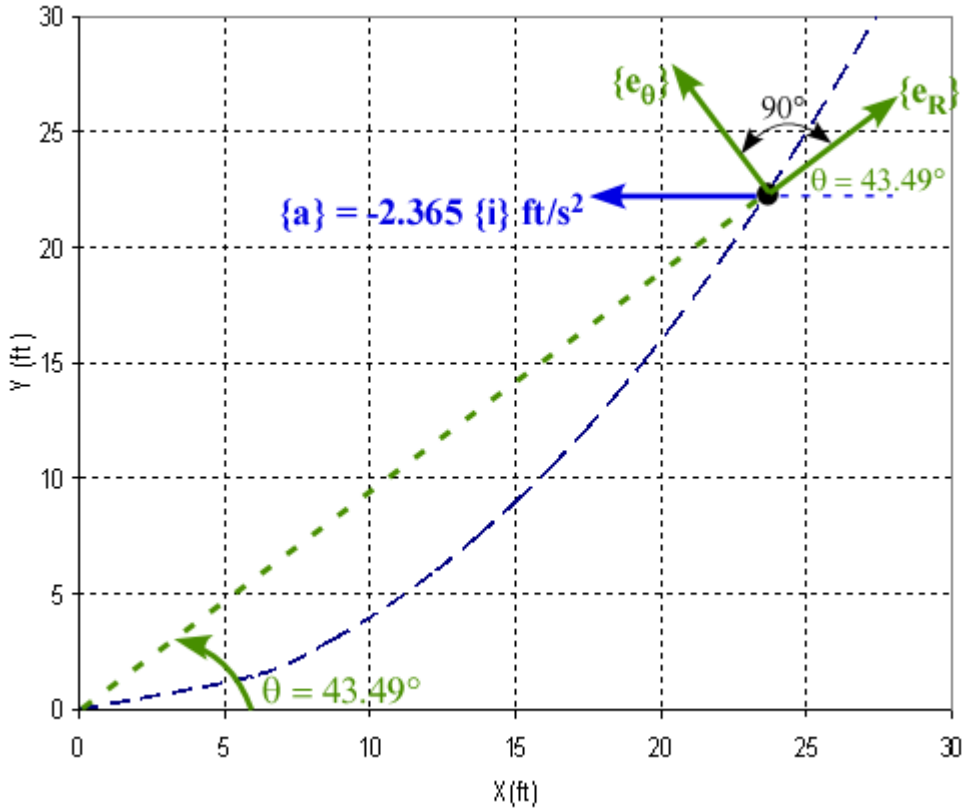
$$\|\{a\}\| = \sqrt{(a_x)^2 + (a_y)^2} = \sqrt{(-2.365)^2 + (0)^2}\ \text{ft}/\text{s}^2$$

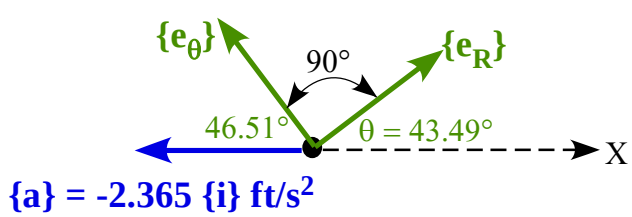
$$\|\{a\}\| = 2.365\ \text{ft}/\text{s}^2$$

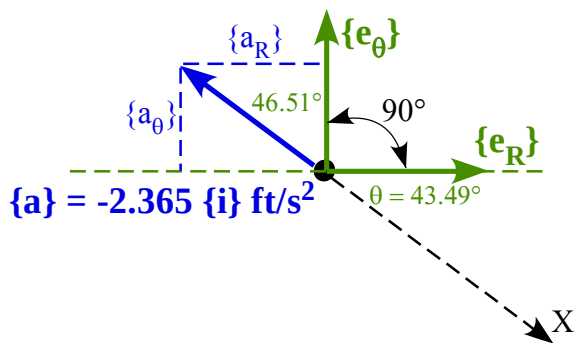
<u>Acceleration vector in Polar:</u> We want to express the SAME acceleration vector {a} but in a different coordinate system. So need locate Polar coordinates – which we have done in Part (a).

In order to express the vector {a} in Polar, we need to find the component directions of {a} along $\{e_R\}$ and $\{e_\theta\}$. We can do this the same 2 ways as for velocity – but since the full acceleration vector lies along the x-direction, they are identical approaches.

So looking at the figure with the acceleration vector and unit vectors for Polar, we can determine the geometry needed.







So solving for the components in Polar:

$$\{a_R\} = \left(-\|\{a\}\|\cos(\theta)\right)\{e_R\}$$

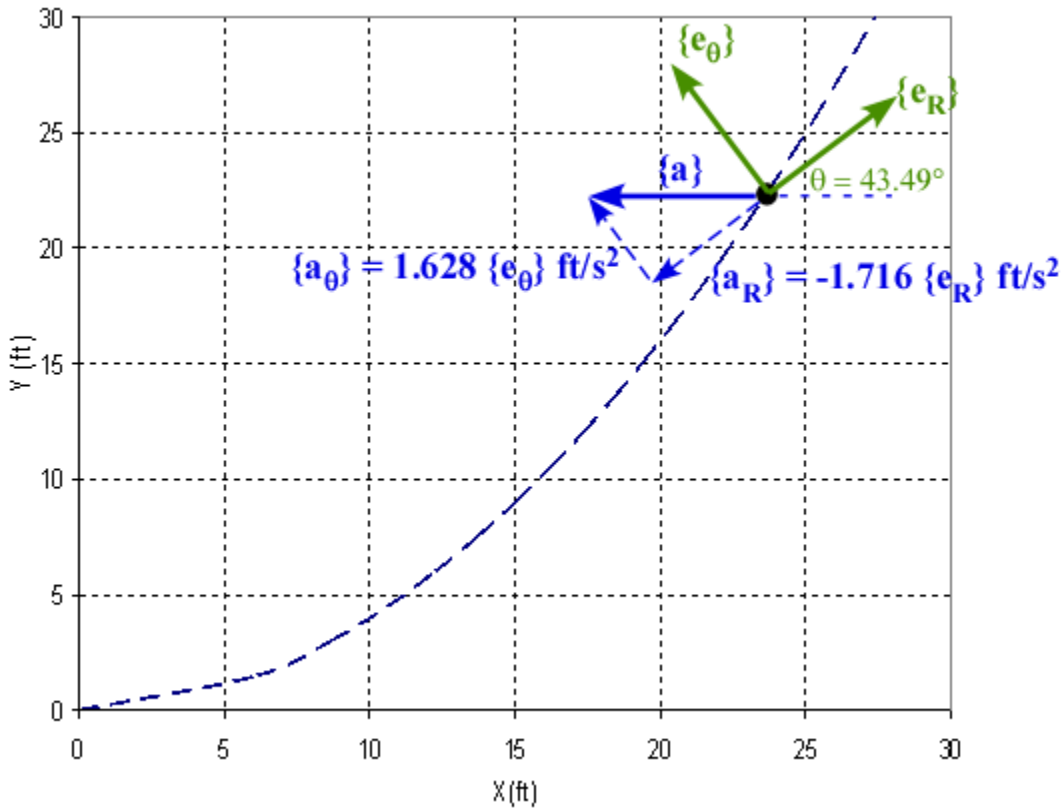
$$= \left(-|-2.365|\cos(43.49)\right)\{e_R\}\ \mathrm{ft}/\mathrm{s}^2$$

$$= -1.716\{e_R\}\ \mathrm{ft}/\mathrm{s}^2$$

$$\{a_\theta\} = \left(\|\{a\}\|\sin(\theta)\right)\{e_\theta\}$$
$$= \left(|-2.365|\sin(43.49)\right)\{e_\theta\}\ \text{ft}/\text{s}^2$$
$$= 1.628\{e_\theta\}\ \text{ft}/\text{s}^2$$

Combining components to express acceleration vector in Polar:

$$\{a\} = \{a_R\} + \{a_\theta\}$$
$$\boxed{\{a\} = -1.716\{e_R\} + 1.628\{e_\theta\}\ \text{ft}/\text{s}^2}$$

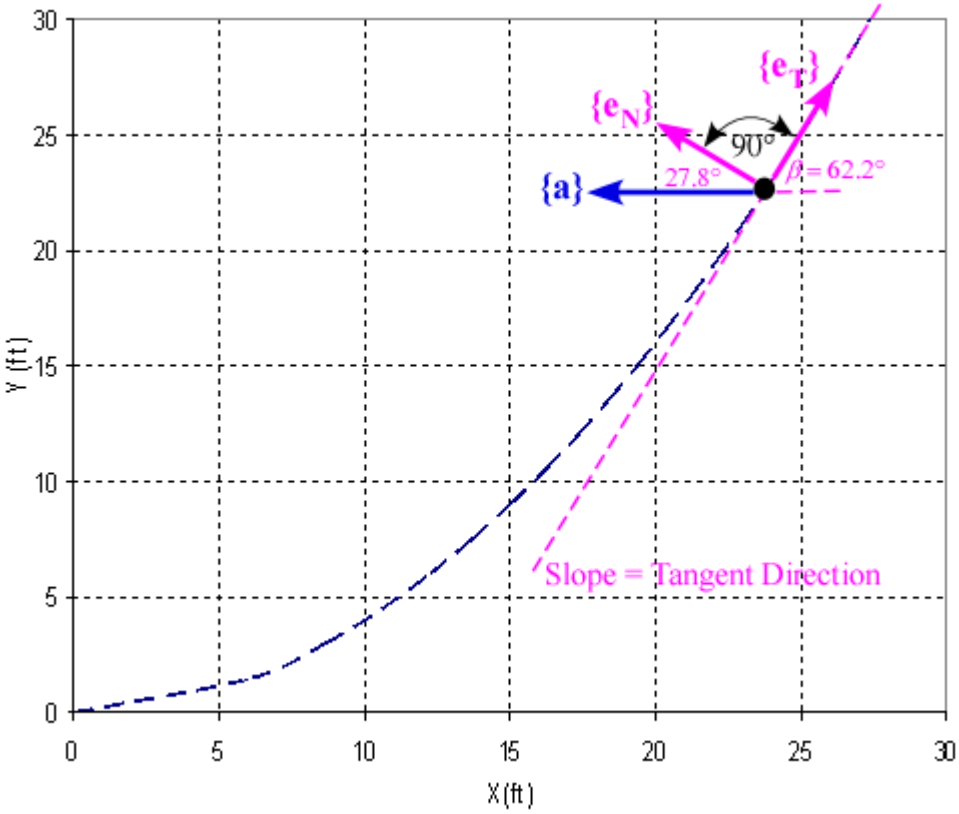


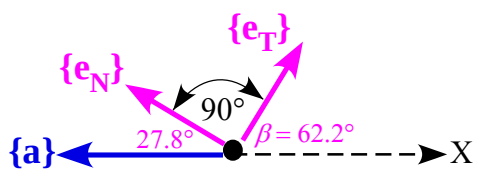
Check magnitude of acceleration:

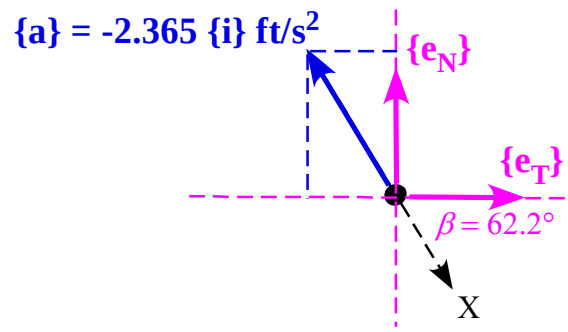
$$\|\{a\}\| = \sqrt{(a_R)^2 + (a_\theta)^2}$$
$$\|\{a\}\| = \sqrt{(-1.716)^2 + (1.628)^2}\ \text{ft}/\text{s}^2$$
$$\|\{a\}\| = 2.365\ \text{ft}/\text{s}^2$$

Checks! Matches magnitude found earlier.

<u>Acceleration vector in Path:</u> We want to express the SAME acceleration vector {a} but in a different coordinate system. So need locate Path coordinates – which we have done in Part (c).







Determining components:

$$\{a_T\} = \left(-\|\{a\}\|\cos(\beta)\right)\{e_T\}\ \mathrm{ft}/\mathrm{s}^2$$

$$\{a_N\} = \left(\|\{a\}\|\sin(\beta)\right)\{e_N\}\ \mathrm{ft}/\mathrm{s}^2$$

Substituting values:

$$\{a_T\} = \left(-|-2.365|\cos\left(62.2^\circ\right)\right)\{e_T\}\ \mathrm{ft}/\mathrm{s}^2$$

$$= -1.103\{e_T\}\ \mathrm{ft}/\mathrm{s}^2$$

$$\{a_N\} = \left(|-2.365|\sin\left(62.2^\circ\right)\right)\{e_N\}\ \mathrm{ft}/\mathrm{s}^2$$

$$= 2.092\{e_N\}\ \mathrm{ft}/\mathrm{s}^2$$

Note that the tangential component of acceleration is positive. This ALWAYS must be true given our definition of Path Coordinates. Combining the components:

$$\{a\} = \{a_T\} + \{a_N\}$$

$$\boxed{\{a\} = -1.103\{e_T\} + 2.092\{e_N\}\ \mathrm{ft}/\mathrm{s}^2}$$

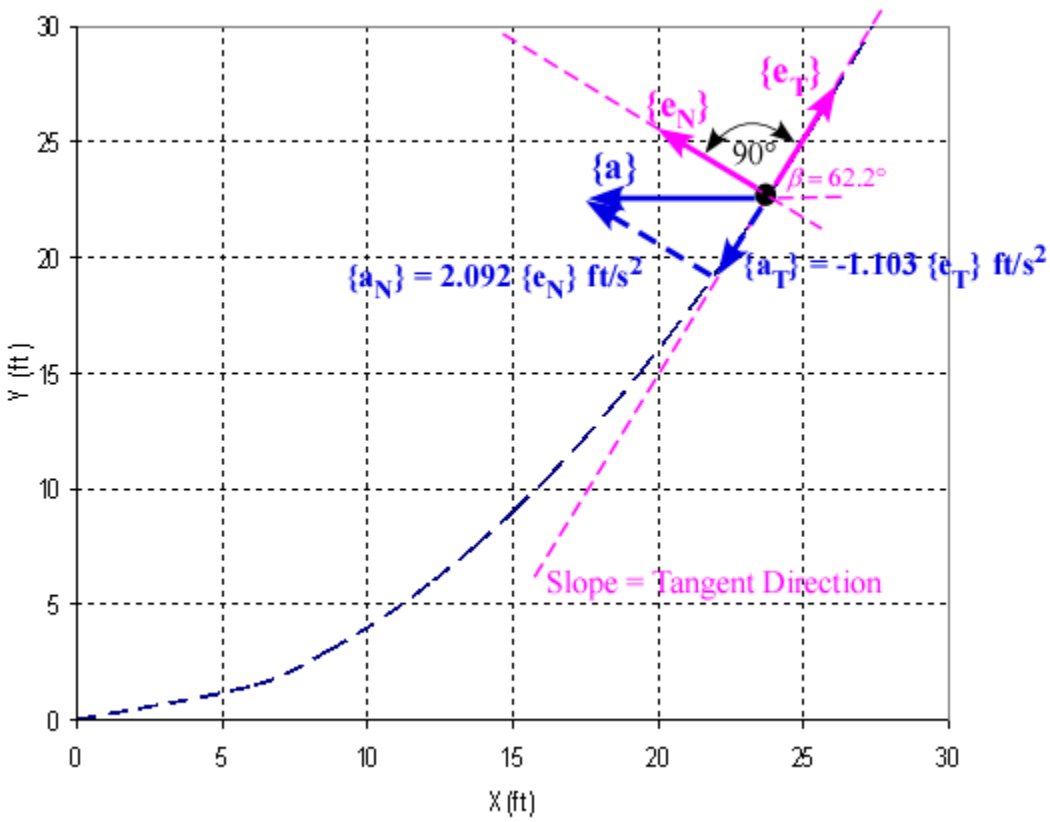
Check magnitude:

$$\|\{a\}\| = \sqrt{\left(\|\{a_T\}\|\right)^2 + \left(\|\{a_N\}\|\right)^2}$$

$$\|\{a\}\| = \sqrt{(a_T)^2 + (a_N)^2}$$

$$\|\{a\}\| = \sqrt{(-1.103)^2 + (2.092)^2}\ \mathrm{ft}/\mathrm{s}^2$$

$$\|\{a\}\| = 2.365\ \mathrm{ft}/\mathrm{s}^2$$

Checks! Matches magnitude found earlier

Graphically: