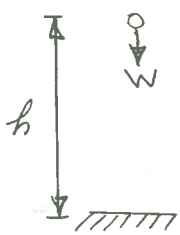
Problem 1: (25 pts) A weight is dropped down a mine shaft and is heard striking the bottom *T* seconds later.

a) If the speed of sound is *c*, find the depth *h* of the shaft as a function of *c*, *T*, and the gravitational constant *g*. *Hint*: break up the problem into two events: (1) weight dropping, and (2) sound traveling.

b) If c = 1126 ft/sec and T = 10 sec, what is the numerical value of the depth *h*?

c) If the finite velocity of sound were ignored (in other words, you assumed that you heard the strike at the exact same time actually hit the bottom), would the depth be overestimated or underestimated? By how much?

Solution



drop portion: a = constant

derive const. accel. equations

$$\int_{v_0}^{v} dv = \int_0^t a\,dt$$

$$v = v_0 + at$$

$$\int_{x_0}^{x} dx = \int_0^{t_1} v\,dt = \int_0^{t_1} (v_0 + at)\,dt$$

$$(x - x_0) = v_0 t_1 + \tfrac{1}{2} a t_1^2$$

for this portion of motion:

$$(x - x_0) = h$$

$$v_0 = 0$$

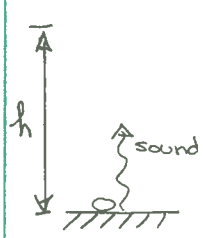
$$a = g$$

So: $h = \frac{1}{2} g t_1^2$ where t = time weight hits bottom

$t_1 = \sqrt{2h/g}$

Sound travel after impact: constant velocity

zero acceleration



$$(x - x_0) = h$$

$$v_0 = c$$

$$a = 0$$

So $h = ct_2 \quad \therefore \quad t_2 = \frac{h}{c}$

For entire motion: $t_1 + t_2 = T$

Substituting: $\sqrt{\frac{2h}{g}} + \frac{h}{c} = T$

$$\left(\sqrt{\frac{2h}{g}}\right)^2 = \left(T - \frac{h}{c}\right)^2$$

$$\frac{2h}{g} = T^2 - 2\frac{Th}{c} + \frac{h^2}{c^2}$$

$$2hc^2 = gc^2T^2 - 2Tgch + gh^2$$

$$0 = gh^2 - (2c^2 + 2gcT)h + gc^2T^2$$

quadratic equation - solve for h

$$h = \frac{(2c^2 + 2gcT) \pm \sqrt{(2c^2+2gcT)^2 - 4(g)(gc^2T^2)}}{2g}$$

$$= \frac{c^2}{g} + cT \pm \left[\frac{4c^4 + 8gc^3T + 4g^2c^2T^2 - 4g^2c^2T^2}{4g^2}\right]^{\frac{1}{2}}$$

$$= \frac{c^2}{g}\left(1 + \frac{gT}{c}\right) \pm \left[\frac{c^4 + 2gc^3T + \cancel{g^2c^2T^2} - \cancel{g^2c^2T^2}}{g^2}\right]^{\frac{1}{2}}$$

$$= \frac{c^2}{g}\left(1 + \frac{gT}{c}\right) \pm \left(\frac{c^4}{g^2} + \frac{2gc^4T}{g^2c}\right)^{\frac{1}{2}}$$

$$= \frac{c^2}{g}\left(1 + \frac{gT}{c}\right) \pm \left[\left(\frac{c^4}{g^2}\right)\left(1 + \frac{2gT}{c}\right)\right]^{\frac{1}{2}}$$

$$= \frac{c^2}{g}\left(1 + \frac{gT}{c}\right) \pm \frac{c^2}{g}\left(1 + \frac{2gT}{c}\right)^{\frac{1}{2}}$$

$$h = \frac{c^2}{g}\left(1 + \frac{gT}{c} \pm \sqrt{1 + \frac{2gT}{c}}\right)$$

b) if $c = 1126\ ft/s$ and $T = 10$ sec, what is h?

$$h = \frac{(1126)^2}{32.2}\left(1 + \frac{(32.2)(10)}{(1126)} \pm \sqrt{1 + \frac{2(32.2)(10)}{1126}}\right)$$

$h = 100002$ ft or $1{,}267.85$ ft

Take smaller root

So well is $\underline{\underline{1268\ ft}}$

c) If speed of sound neglected: then $T = t_1$

$$t_1 = \sqrt{2g/h} \rightarrow t_1^2 g/2 = h$$

$$h = (10)^2(32.2)/2$$

$$h = \underline{\underline{1610\ ft}}$$

So we would overestimate the depth.