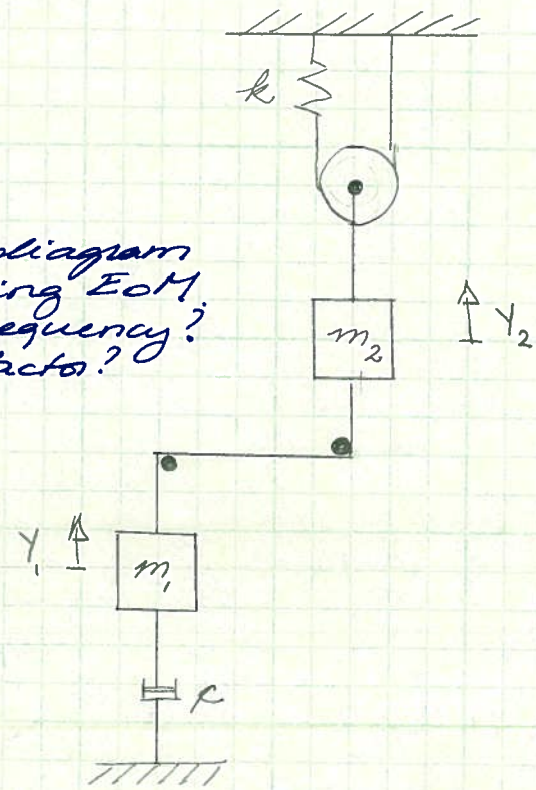
Problem 3.24 //



a) Draw a free-body diagram & derive SINGLE governing EoM. What is the natural frequency? What is the damping factor?

b) Remove the damper & derive the equation of motion using conservation of energy

<u>Case I</u>: Y_i = displacement of center of mass of ith mass from its static equilibrium position

Positive motion as shown above

Kinematic Constraint Equations

1) $Y_1 = Y_2$

2) Y_s = elongation of spring due to dynamic motion

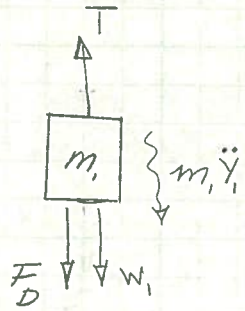
$Y_s = 2Y_2$

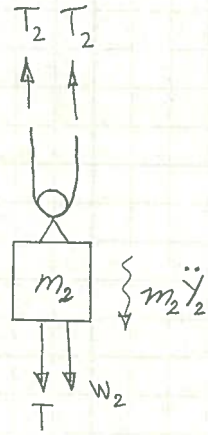
<u>OPTION (1) — use total spring forces</u>

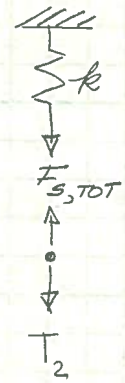
Define: Y_{s0} = elongation of spring due to static/gravity load = const. value

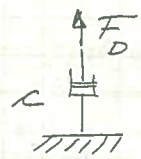
Note: you should be able to solve for Y_{s0} as it is solution to purely static problem → ENGR 221

Dynamic FBD's - since using total spring forces, must include gravity









$F_D = c\dot{y}_1$

$F_{S,TOT}$ = total spring force

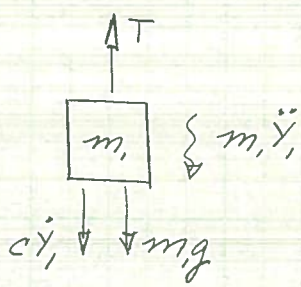
$= k(y_{s0} - y_s)$

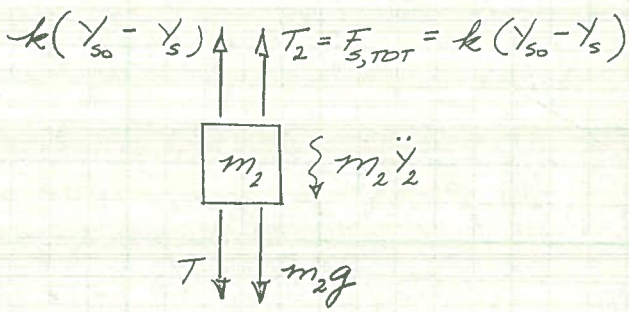
Note that as m_2 moves in (+) direction, the total force on spring is reduced.

$W_1 = m_1 g$

$W_2 = m_2 g$

Substituting above eqns & only drawing FBD's of masses:





Apply equilibrium:

Mass 1: $\Sigma F = 0$ $\quad m_1\ddot{y}_1 + m_1 g + c\dot{y}_1 - T = 0$

Mass 2: $\Sigma F = 0$ $\quad m_2\ddot{y}_2 + m_2 g + T - 2k(y_{s0} - y_s) = 0$

Combining:

$$m_2\ddot{y}_2 + m_2 g + (m_1\ddot{y}_1 + m_1 g + c\dot{y}_1) - 2k(y_{s0} - y_s) = 0$$

$$m_2\ddot{y}_2 + m_1\ddot{y} + (m_2 + m_1)g + c\dot{y}_1 - 2ky_{s0} + 2ky_s = 0$$

Rearranging so that static terms next to one another

$$(m_2\ddot{y}_2 + m_1\ddot{y}_1 + c\dot{y}_1 + 2ky_s) + \underbrace{\cancel{(m_2 + m_1)g - 2ky_{s0}}}_{\text{draw static FBD to convince yourself terms cancel}} = 0$$

So:

$$m_2\ddot{y}_2 + m_1\ddot{y}_1 + c\dot{y}_1 + 2ky_s = 0$$

Substituting kinematic constraints:

$$m_2\ddot{y}_2 + m_1\ddot{y}_2 + c\dot{y}_2 + 2k(2y_2) = 0$$

$$\boxed{(m_2 + m_1)\ddot{y}_2 + c\dot{y}_2 + (4k)y_2 = 0}$$

Equivalent mass: $m^* = m_1 + m_2$

Equivalent damping: $c^* = c$

Equivalent stiffness: $k^* = 4k$

Natural frequency: $\boxed{\omega_n = \sqrt{\frac{k^*}{m^*}} = \sqrt{\frac{4k}{m_1 + m_2}} = 2\sqrt{\frac{k}{m_1 + m_2}}}$

Damping Ratio: $\zeta = \frac{c^*}{2m^*\omega_n} = \frac{c}{2(m_1 + m_2)\left(2\sqrt{\frac{k}{m_1 + m_2}}\right)}$

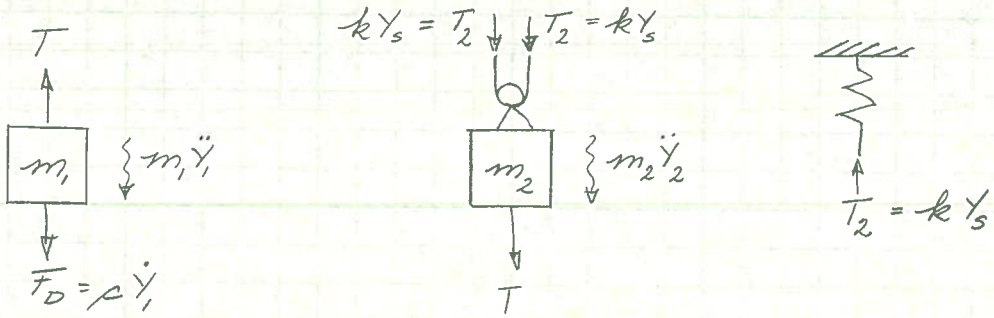
$$\boxed{\zeta = \frac{c}{4\sqrt{(m_1 + m_2)k}}}$$

OPTION (2) - use incremental/dynamic spring forces

Can use this approach when you can "see" that, since measuring motion about equilibrium point, all static/gravity terms will cancel.

Kinematic constraint equations still valid
$\Rightarrow$ same co-ordinate system

Dynamic FBD's



Apply equilibrium:

Mass 1: $m_1\ddot{Y}_1 + c\dot{Y}_1 - T = 0$

Mass 2: $m_2\ddot{Y}_2 + T + 2kY_s = 0$

Combining:

$$m_1\ddot{Y}_1 + m_2\ddot{Y}_2 + 2kY_s + c\dot{Y}_1 = 0$$

Substituting kinematic constraint equations

$$\boxed{(m_1 + m_2)\ddot{Y}_2 + c\dot{Y}_1 + (4k)Y_2 = 0}$$

So:

$$\omega_n = \sqrt{\frac{k^*}{m^*}} = 2\sqrt{\frac{k}{m_1 + m_2}}$$

Note: resulting eqn is EXACTLY the same. Should be the case for same coordinate syst.

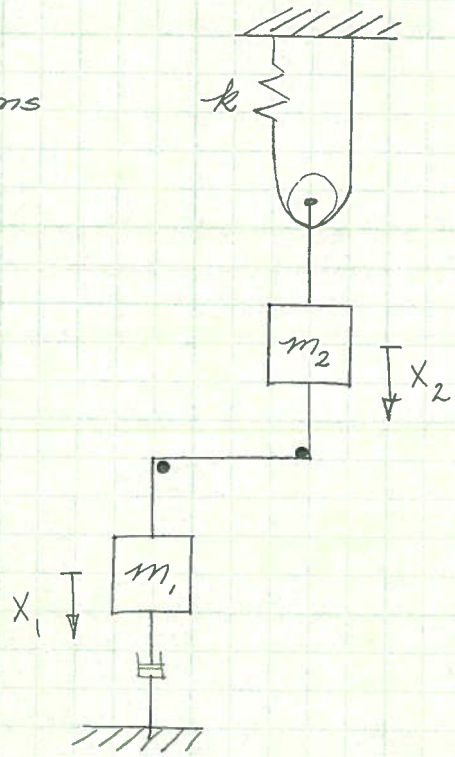
$$\zeta = \frac{c^*}{2m^*\omega_n} = \frac{c}{4\sqrt{(m_1+m_2)k}}$$

Case II :

X_i = displacement of center of mass of ith mass such that:

$X_i = 0$ when spring undeformed $\forall i$

Positive directions

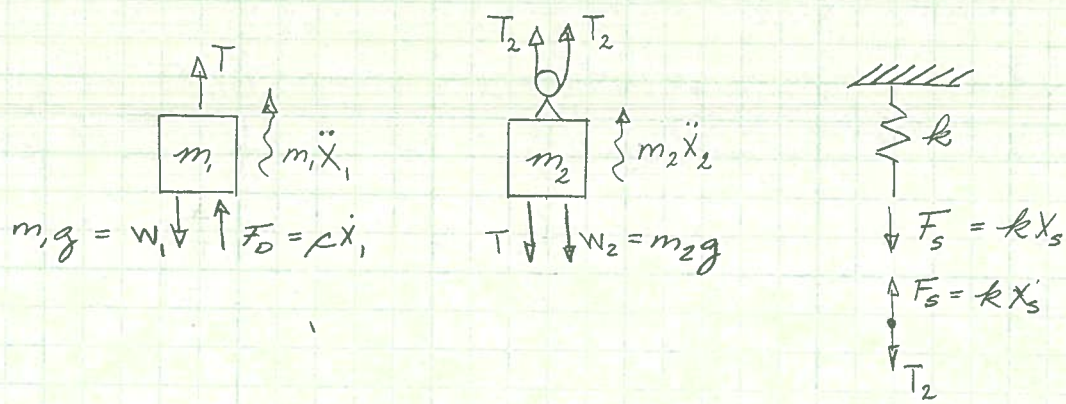


X_s = total elongation of spring from undeformed pos.

Kinematic Constraints:

- $X_1 = X_2$
- $X_s = 2X_2$

Dynamic FBD's → use TOTAL forces



Mass 1: $\Sigma F = 0 \rightarrow T + m_1\ddot{X}_1 + c\dot{X}_1 = m_1 g$

Mass 2: $\Sigma F = 0 \rightarrow m_2\ddot{X}_2 + 2T_2 - T = m_2 g$

$$m_2\ddot{X}_2 + 2(kX_s) - T = m_2 g$$

Combining Eqns:

$$m_2 \ddot{x}_2 + 2(kx_s) + m_1 \ddot{x}_1 + c\dot{x}_1 = W_1 + W_2$$

Substituting Kinematic Constraints

$$\boxed{(m_2 + m_1)\ddot{x}_2 + c\dot{x}_2 + (4k)x_2 = m_1 g + m_2 g}$$

Equivalent mass: $m^* = m_1 + m_2$

Equivalent damping: $c^* = c$

Equivalent stiffness: $k^* = 4k$

Natural frequency $\boxed{\omega_n = \sqrt{\frac{k^*}{m^*}} = 2\sqrt{\frac{k}{m_1 + m_2}}}$

Damping Ratio: $\boxed{\zeta = \frac{c^*}{2m^*\omega_n} = \frac{c}{4\sqrt{(m_1 + m_2)k}}}$

Note that, though your EoM is different depending on degree of freedom chosen, the dynamic properties (frequency & damping) do NOT change.

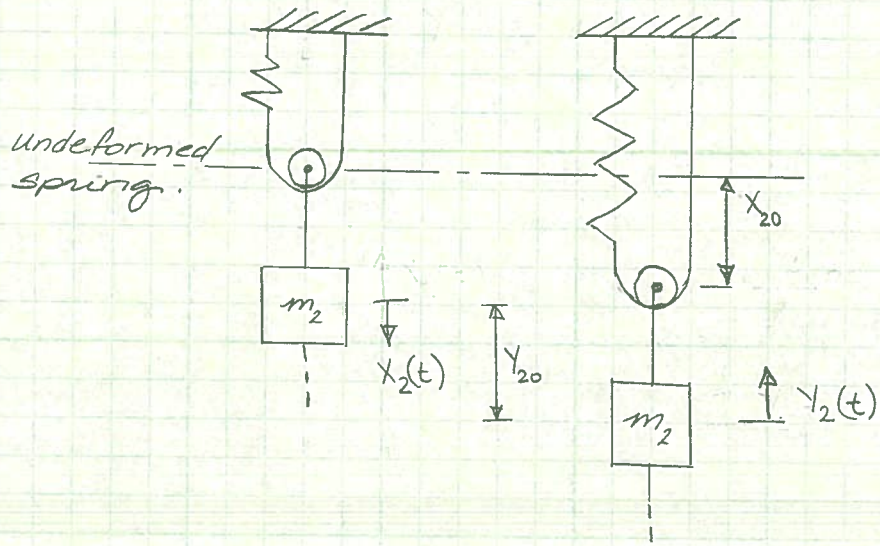
If you had taken positive direction to be upwards then resulting EoM would be:

$$(m_1 + m_2)\ddot{x}_2 + c\dot{x}_2 + (4k)x_2 = -m_1 g - m_2 g$$

EXTRA: How can you tell that both expressions of EOM are equivalent?

Show that can go from one equation to the other.

In order to transform from 1 coord. system to the other, need to define kinematic relationship



$$X_{20} = \frac{1}{2} Y_{50}$$

So: $X_2(t) = Y_{20} - Y_2(t)$

Taking derivaties: $\dot{X}_2 = -\dot{Y}_2$

$$\ddot{X}_2 = -\ddot{Y}_2$$

Substituting into: $(m_1+m_2)\ddot{X}_2 + c\dot{X}_2 + (4k)X_2 = W_1 + W_2$

$$(m_1+m_2)(-\ddot{Y}_2) - c(\dot{Y}_2) + (4k)(X_{20} - Y_2(t)) = W_1 + W_2$$

$$-(m_1+m_2)\ddot{Y}_2 - c\dot{Y}_2 - (4k)Y_2 + 4kX_{20} - W_1 - W_2 = 0$$

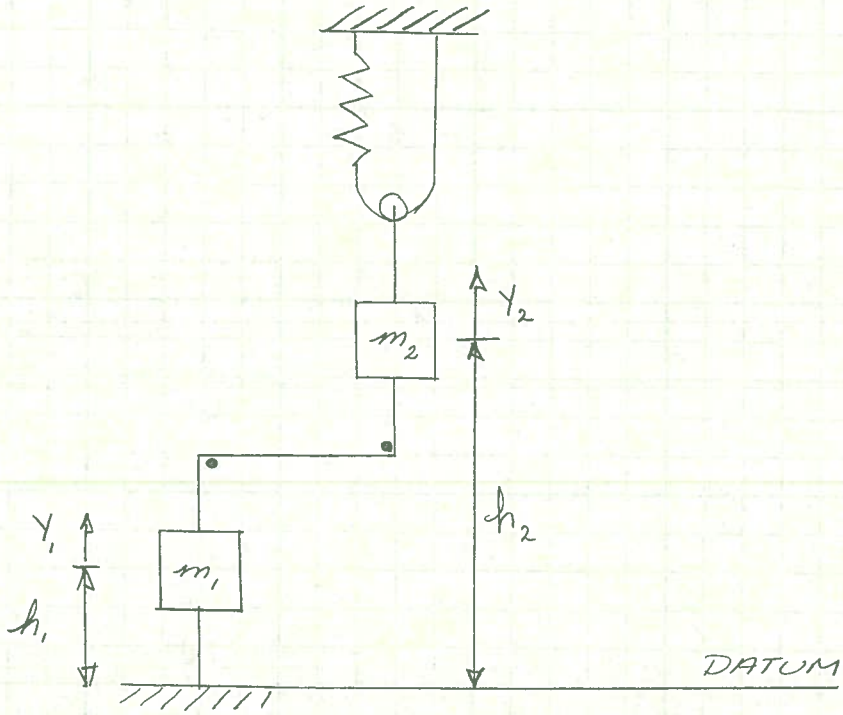
$$-(m_1+m_2)\ddot{Y}_2 - c\dot{Y}_2 - 4kY_2 \underbrace{+ \cancel{2kY_{50}} - \cancel{m_1 g} - \cancel{m_2 g}}_{\text{from static equilibrium}} = 0$$

Multiply eqn by -1

$$\boxed{(m_1+m_2)\ddot{Y}_2 + c\dot{Y}_2 + 4kY_2 = 0}$$

✓

b) Derive equation using energy equilibrium
Neglect damping



<u>Case I</u>: Measuring from equilibrium

<u>OPTION (1)</u>: Use total energy term

$$T_o + V_o = \text{constant}$$

$$T = \tfrac{1}{2} m_1 \dot{Y}_1^2 + \tfrac{1}{2} m_2 \dot{Y}_2^2$$

$$V = m_1 g (h_1 + Y_1) + m_2 g (h_2 + Y_2) + \tfrac{1}{2} k (Y_{so} - Y_s)^2$$

From conservation: $T_o + V_o = T + V$

$$\text{CONST.} = \tfrac{1}{2}(m_1 + m_2)\dot{Y}_2^2 + (m_1 + m_2) g Y_2 + \underbrace{m_1 g h_1 + m_2 g h_2}_{\text{constant}} + \tfrac{1}{2} k (Y_{so} - Y_s)^2$$

Taking derivative:

$$0 = (m_1 + m_2)\dot{Y}_2 \ddot{Y}_2 + (m_1 + m_2) g \dot{Y}_2 + k (Y_{so} - Y_s)(-\dot{Y}_s)$$

$$0 = (m_1 + m_2)\dot{Y}_2 \ddot{Y}_2 + (m_1 + m_2) g \dot{Y}_s - k Y_{so} \dot{Y}_s + k Y_s \dot{Y}_s$$

Substituting: $Y_s = 2Y_2$

$\dot{Y}_s = 2\dot{Y}_2$

$$0 = (m_1+m_2)\dot{Y}_2\ddot{Y}_2 + (m_1+m_2)g(2\dot{Y}_2) - kY_{so}(2\dot{Y}_2) + k(2Y_2)(2\dot{Y}_2)$$

Cancel $\dot{Y}_2$ terms

$$0 = (m_1+m_2)\ddot{Y}_2 + \underbrace{\cancel{2(m_1+m_2)g - 2kY_{so}}}_{\text{From statics, these terms cancel}} + 4kY_2$$

Resulting Equation

$$\boxed{(m_1+m_2)\ddot{Y}_2 + 4kY_2 = 0}$$

Same as part (a) w/o damping term ✓

<u>OPTION (2)</u>: Use change in energy due to dynamics

$$T_0 + V_0 = 0$$

$$T = \tfrac{1}{2}m_1\dot{Y}_1^2 + \tfrac{1}{2}m_2\dot{Y}_2^2$$

$$V = \tfrac{1}{2}kY_s^2$$

$$\tfrac{1}{2}m_1\dot{Y}_1^2 + \tfrac{1}{2}m_2\dot{Y}_2^2 + \tfrac{1}{2}kY_s^2 = 0$$

$$\tfrac{1}{2}(m_1+m_2)\dot{Y}_2^2 + \tfrac{1}{2}k(2Y_2)^2 = 0$$

Taking derivative:

$$(m_1+m_2)\dot{Y}_2\ddot{Y}_2 + k(2Y_2)(2\dot{Y}_2) = 0$$

Cancelling $\dot{Y}_2$

$$\boxed{(m_1+m_2)\ddot{Y}_2 + 4kY_2 = 0}$$ ✓

Case II

$X_i \Rightarrow$ measure from undeformed spring position

$$T_o + V_o = \text{constant}$$

$$T = \frac{1}{2} m_1 \dot{X}_1^2 + \frac{1}{2} m_2 \dot{X}_2^2 = \frac{1}{2}(m_1 + m_2)\dot{X}_2^2$$

$$V = m_1 g (h_1 - X_1) + m_2 g (h_2 - X_2) + \frac{1}{2} k X_s^2$$

$$= m_1 g h_1 + m_2 g h_2 - (m_1 + m_2) g X_2 + \frac{1}{2} k (2X_2)^2$$

$$\text{Const} = \frac{1}{2}(m_1 + m_2)\dot{X}_2^2 + m_1 g h_1 + m_2 g h_2 - (m_1 + m_2) g X_2 + 2k X_2^2$$

Taking derivative

$$0 = (m_1 + m_2)\dot{X}_2 \ddot{X}_2 - (m_1 + m_2) g \dot{X}_2 + 4k X_2 \dot{X}_2$$

Cancel out $\dot{X}_2$

$$0 = (m_1 + m_2)\ddot{X}_2 - (m_1 + m_2) g + 4k X_2$$

Rearranging:

$$(m_1 + m_2)\ddot{X}_2 + 4k X_2 = m_1 g + m_2 g$$

$$\boxed{(m_1 + m_2)\ddot{X}_2 + 4k X_2 = W_1 + W_2}$$

✓ Same as part (a) for this degree-of-freedom

c) Find EOM using Lagrange's Equation
Neglect damper in system

<u>CASE I</u>: dof measured from equilibrium

OPTION (1)

$$T = \frac{1}{2} m_1 \dot{Y}_1^2 + \frac{1}{2} m_2 \dot{Y}_2^2 = \frac{1}{2} m_1 \dot{Y}_2^2 + \frac{1}{2} m_2 \dot{Y}_2^2 = \frac{1}{2}(m_1+m_2)\dot{Y}_2^2$$

$$V = m_1 g (h_1 + Y_1) + m_2 g (h_2 + Y_2) + \frac{1}{2} k (Y_{so} - Y_s)^2$$
$$= m_1 g (h_1 + Y_2) + m_2 g (h_2 + Y_2) + \frac{1}{2} k (Y_{so} - 2Y_2)^2$$

$$\frac{\partial T}{\partial Y_2} = 0$$

$$\frac{\partial T}{\partial \dot{Y}_2} = \frac{1}{2}(m_1+m_2)[2\dot{Y}_2] = (m_1+m_2)\dot{Y}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}_2}\right) = \frac{d}{dt}\left((m_1+m_2)\dot{Y}_2\right) = (m_1+m_2)\ddot{Y}_2$$

To simplify V term, I'll expand out expression

$$V = m_1 g h_1 + m_1 g Y_2 + m_2 g h_2 + m_2 g Y_2 + \frac{1}{2} k [Y_{so}^2 - 4Y_{so}Y_2 + 4Y_2^2]$$

So: $$\frac{\partial V}{\partial Y_2} = m_1 g + m_2 g + \frac{1}{2} k [-4Y_{so} + 8Y_2]$$

$$= \underbrace{(m_1+m_2)g - 2kY_{so}}_{\text{from statics, terms cancel}} + 4Y_2$$

Substituting:

$$(m_1+m_2)\ddot{Y}_2 - 0 + 4Y_2 + 0 = 0$$

$$\boxed{(m_1+m_2)\ddot{Y}_2 + 4Y_2 = 0}$$

Agrees w/ results of part (b)

OPTION (2):

$$T = \frac{1}{2}(m_1+m_2)\dot{Y}_2^2$$

$$V = \frac{1}{2}kY_s^2 = \frac{1}{2}k(2Y_2)^2 = 2kY_2^2$$

Then: $\frac{\partial T}{\partial Y_2} = 0$

$$\frac{\partial T}{\partial \dot{Y}_2} = (m_1+m_2)\dot{Y}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{Y}_2}\right) = (m_1+m_2)\ddot{Y}_2$$

$$\frac{\partial V}{\partial Y_2} = 4kY_2$$

So:

$$(m_1+m_2)\ddot{Y}_2 + 4kY_2 = 0$$

CASE II: X_i measured from undeformed spring

$$T = \frac{1}{2}(m_1+m_2)\dot{X}_2^2$$

$$V = m_1gh_1 + m_2gh_2 - (m_1+m_2)gX_2 + 2kX_2^2$$

$$\partial T/\partial X_2 = 0 \qquad \frac{\partial V}{\partial X_2} = -(m_1+m_2)g + 4kX_2$$

$$\partial T/\partial \dot{X}_2 = (m_1+m_2)\dot{X}_2 \qquad \frac{d}{dt}\left(\partial T/\partial \dot{X}_2\right) = \frac{d}{dt}\left[(m_1+m_2)\dot{X}_2\right] = (m_1+m_2)\ddot{X}_2$$

Substitute:

$$(m_1+m_2)\ddot{X}_2 - (m_1+m_2)g + 4kX_2 = 0$$

$$\boxed{(m_1+m_2)\ddot{X}_2 + 4kX_2 = (m_1+m_2)g}$$

Again, agrees w/ previous.