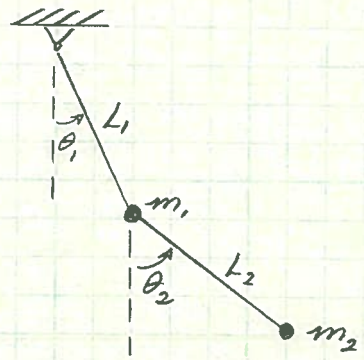
Double Pendulum



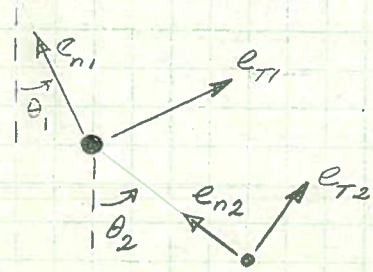
Derive EoM using equilib. and Lagrange's. Discuss differences & show how they can be reconciled.

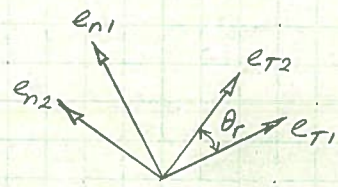
Newton's → need accelerations

Lagranges → need velocities

θ_1 = total rotation of m_1-bar from vertical
θ_2 = Relative rotation of m_2-bar from vertical defined by position of m_1

Coordinate System - It will be convenient to use normal & tangential components. Note that since path is a circle, will look very much like polar coords





$\theta_r = \theta_2 - \theta_1$

The two coordinate systems are related through:

$$\begin{Bmatrix} e_{T1} \\ e_{N1} \end{Bmatrix} = \begin{bmatrix} \cos(\theta_2-\theta_1) & -\sin(\theta_2-\theta_1) \\ \sin(\theta_2-\theta_1) & \cos(\theta_2-\theta_1) \end{bmatrix} \begin{Bmatrix} e_{T2} \\ e_{N2} \end{Bmatrix}$$

clockwise pot.

where: $\{e_{T1}\}$ = unit vector in tangential direction to motion of m_1

$\{e_{N1}\}$ = unit vector in normal direction to motion of m_2

Free-Body Diagrams

where: a_{1T} = acceleration of m_1 in tangential direction (along $\{e_{T1}\}$)
a_{1N} = " " m_1 in normal direction (along $\{e_{N1}\}$)

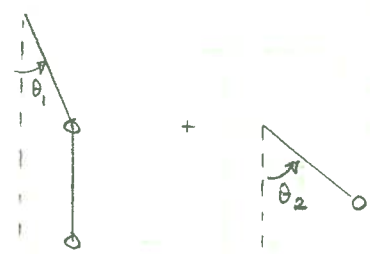
a_{2T} = " " m_2 in tangential " $\{e_{T2}\}$
a_{2N} = " " m_2 " normal $\{e_{N2}\}$

Velocity Expressions - needed for Lagrange's

$\{V_1\}$ = velocity of mass m_1 $= (L_1\dot{\theta}_1)\{e_{T1}\} = v_{T1}\{e_{T1}\}$

$\{V_2\}$ = velocity of mass m_2 $= \{V_1\} + (L_2\dot{\theta}_2)\{e_{T2}\}$

$= (L_1\dot{\theta}_1)\{e_{T1}\} + (L_2\dot{\theta}_2)\{e_{T2}\}$



Note that $\{V_2\}$ is written in terms of 2 coordinate systems
Use transformation to re-write:

$$\{V_2\} = (L_1\dot{\theta}_1)\left[\cos(\theta_2-\theta_1)\{e_{T2}\} - \sin(\theta_2-\theta_1)\{e_{N2}\}\right] + (L_2\dot{\theta}_2)\{e_{T2}\}$$

$$= \underbrace{(L_1\dot{\theta}_1\cos(\theta_2-\theta_1) + L_2\dot{\theta}_2)}_{\substack{\text{magnitude of velocity in tangential} \\ = v_{T2}}}\{e_{T2}\} \underbrace{- L_1\dot{\theta}_1\sin(\theta_2-\theta_1)}_{\substack{\text{magnitude of velocity in normal} \\ = v_{N2}}}\{e_{N2}\}$$

To find total velocity magnitudes:

$$\|\{V_1\}\| = \sqrt{v_{T1}^2} \quad ; \quad \|\{V_2\}\| = \sqrt{v_{T2}^2 + v_{N2}^2}$$

Acceleration Expressions - needed for Newton's

$$\{a_1\} = (L_1\ddot{\theta}_1)\{e_{T1}\} + (L_1\dot{\theta}_1^2)\{e_{N1}\}$$

$$\{a_2\} = \{a_1\} + (L_2\ddot{\theta}_2)\{e_{T2}\} + (L_2\dot{\theta}_2^2)\{e_{N2}\}$$

Again, use coordinate transformation to get $\{a_2\}$ in only one coord. system:

$$\{a_2\} = (L_1\ddot{\theta}_1)\left[\cos(\theta_2-\theta_1)\{e_{T2}\} - \sin(\theta_2-\theta_1)\{e_{N2}\}\right]$$
$$+ (L_1\dot{\theta}_1^2)\left[\sin(\theta_2-\theta_1)\{e_{T2}\} + \cos(\theta_2-\theta_1)\{e_{N2}\}\right]$$
$$+ (L_2\ddot{\theta}_2)\{e_{T2}\} + (L_2\dot{\theta}_2^2)\{e_{N2}\}$$

Grouping:

$$\{a_2\} = \left[L_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_2-\theta_1) + L_2\ddot{\theta}_2\right]\{e_{T2}\}$$
$$+ \left[-L_1\ddot{\theta}_1\sin(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\cos(\theta_2-\theta_1) + L_2\dot{\theta}_2^2\right]\{e_{N2}\}$$
$$= a_{T2}\{e_{T2}\} + a_{N2}\{e_{N2}\}$$

where:

$$a_{T2} = L_1\ddot{\theta}_1\cos(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\sin(\theta_2-\theta_1)\, L_2\ddot{\theta}_2$$
$$a_{N2} = -L_1\ddot{\theta}_1\sin(\theta_2-\theta_1) + L_1\dot{\theta}_1^2\cos(\theta_2-\theta_1) + L_2\dot{\theta}_2^2$$

Now have magnitudes of acceleration in 2 orthogonal directions. Substitute into FBD's & apply equilibrium to get EOM.

LAGRANGE'S

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0$$

No external forces

Kinetic Energy:

$$T = \frac{1}{2} m_1 \left(\|\{V_1\}\|\right)^2 + \frac{1}{2} m_2 \left(\|\{V_2\}\|\right)^2$$

$$= \frac{1}{2} m_1 (v_{T1})^2 + \frac{1}{2} m_2 \left(v_{T2}^2 + v_{N2}^2\right)$$

$$= \frac{1}{2} m_1 v_{T1}^2 + \frac{1}{2} m_2 v_{T2}^2 + \frac{1}{2} m v_{N2}^2$$

$$= \frac{1}{2} m_1 (L_1 \dot{\theta}_1)^2 + \frac{1}{2} m_2 \left(L_1 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + L_2 \dot{\theta}_2\right)^2$$

$$+ \frac{1}{2} m_2 \left(-L_1 \dot{\theta}_1 \sin(\theta_2 - \theta_1)\right)^2$$

Expanding:

$$T = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[L_1^2 \dot{\theta}_1^2 \cos^2(\theta_2 - \theta_1) + L_2^2 \dot{\theta}_2^2 \right.$$

$$\left. + 2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right] + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 \sin^2(\theta_2 - \theta_1)\right)$$

Recall that $\cos^2 + \sin^2 = 1$

$$T = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$= \frac{1}{2} (m_1 + m_2) L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

Potential Energy

$$V = m_1 g L_1 (1 - \cos\theta_1) + m_2 g \left[L_1 (1 - \cos\theta_1) + L_2 (1 - \cos\theta_2) \right]$$

$$= (m_1 + m_2) g L_1 (1 - \cos\theta_1) + m_2 g L_2 (1 - \cos\theta_2)$$

MASS 1 : θ_1

Derivatives:

$$\frac{\partial T}{\partial \theta_1} = \left(m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)(+1) \right)$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = \frac{1}{\cancel{2}}(m_1+m_2) L_1^2 (\cancel{2}\dot{\theta}_1) + \left(m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1) \right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1+m_2) L_1^2 \ddot{\theta}_1 + (m_2 L_1 L_2)\left[\ddot{\theta}_2 \cos(\theta_2-\theta_1) - \dot{\theta}_2 \sin(\theta_2-\theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\right]$$

$$\frac{\partial V}{\partial \theta_1} = \left((m_1+m_2) g L_1\right) \sin\theta_1$$

Substitute: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = 0$

$$(m_1+m_2) L_1^2 \ddot{\theta}_1 + (m_2 L_1 L_2)\left[\ddot{\theta}_2 \cos(\theta_2-\theta_1) - \dot{\theta}_2 (\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2-\theta_1)\right]$$
$$- \left(m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)\right) + (m_1+m_2) g L_1 \sin\theta_1 + m_2 g L_2 \sin\theta_2 = 0$$

$$(m_1+m_2) L_1^2 \ddot{\theta}_1 + (m_2 L_1 L_2)\ddot{\theta}_2 \cos(\theta_2-\theta_1) - (m_2 L_1 L_2)\dot{\theta}_2^2 \sin(\theta_2-\theta_1)$$
$$+ (m_1+m_2) g L_1 \sin(\theta_1) = 0$$

→ Exact non-linear equation

Linearize equation: Assume small $\measuredangle$'s $\sin\theta \approx \theta$, $\cos\theta \approx 1$

Neglect effect higher order terms $\left(\dot{\theta}_2^2(\theta_2-\theta_1)\right)$

$$(m_1+m_2) L_1^2 \ddot{\theta}_1 + (m_2 L_1 L_2)\ddot{\theta}_2 + (m_1+m_2) g L_1 \theta_1 = 0$$

→ Linearized.

= MASS 2 : θ_2 =

Derivatives

$$\frac{\partial T}{\partial \theta_2} = -m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = \frac{1}{\cancel{2}} m_2 L_2^2 (\cancel{2}\dot{\theta}_2) + (m_2 L_1 L_2)\dot{\theta}_1 \cos(\theta_2 - \theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = (m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\left[\ddot{\theta}_1 \cos(\theta_2 - \theta_1) - \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\right]$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g L_2 \sin\theta_2$$

Substituting

$$(m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\left[\ddot{\theta}_2 \cos(\theta_2 - \theta_1) - \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1)\sin(\theta_2 - \theta_1)\right]$$
$$+ m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + m_2 g L_2 \sin\theta_2 = 0$$

$$(m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\ddot{\theta} \cos(\theta_2 - \theta_1) + (m_2 L_1 L_2)\dot{\theta}_1^2 \sin(\theta_2 - \theta_1)$$
$$+ (m_2 g L_2)\sin\theta_2 = 0$$

→ Complete non-linear equation

Linearize equation: Assume small $\measuredangle$'s

$$(m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\ddot{\theta} + (m_2 g L_2)\theta_2 = 0$$

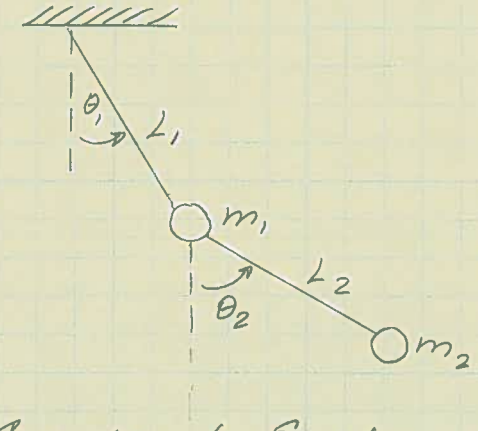
Resulting EoM : Matrix Format

Linearized:

$$\begin{bmatrix} (m_1+m_2)L_1^2 & m_2L_1L_2 \\ m_2L_1L_2 & m_2L_2^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (m_1+m_2)gL_1 & 0 \\ 0 & m_2gL_2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Why can we not write nonlinear EoM in matrix form?
Cross term of $\dot{\theta}_2^2(\theta_2-\theta_1)$ cannot easily be expressed.
Can do it if make system matrix a function of response.

D'Alembert's Principle



Same DOF definitions:

θ_1 = total CCW rotation of m_1-bar from vertical @ support

θ_2 = CCW rotation of m_2-bar RELATIVE to position of m_1

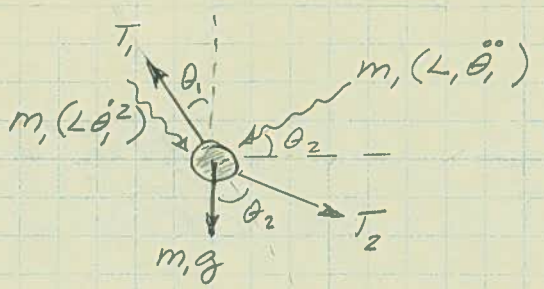
Coordinate System as on page 1

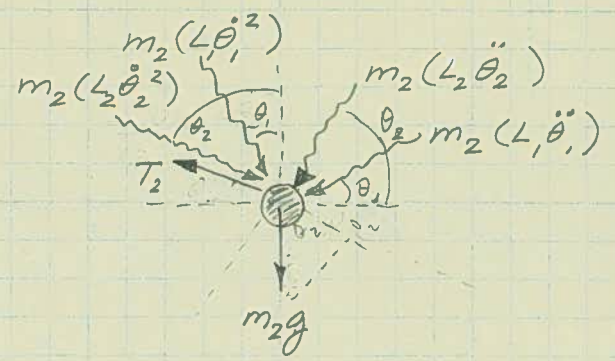
Accelerations: Recall from pg 3

$$\{a_1\} = (L_1\ddot{\theta}_1)\{e_{T1}\} + (L_1\dot{\theta}_1^2)\{e_{N1}\}$$

$$\{a_2\} = \{a_1\} + \underbrace{(L_1\ddot{\theta}_2)\{e_{T2}\} + (L_2\dot{\theta}_2^2)\{e_{N2}\}}_{\text{acceleration of } m_2 \text{ relative to } m_1}$$

FBD's - Total Forces





Note that can just add individual inertial force components onto FBD - do not need to convert to single coordinate system.

Apply Equilibrium

MASS 2 - Take moments about position of m_1

$$(m_2 L_2 \ddot{\theta}_2) L_2 + (m_2 L_1 \ddot{\theta}_1 \cos\theta_1)(L_2 \cos\theta_2)$$
$$+ (m_2 L \ddot{\theta}_1 \sin\theta_1)(L_2 \sin\theta_2) + (m_2 L_1 \dot{\theta}_1^2 \cos\theta_1)(L \sin\theta_2)$$
$$- (m_2 L_1 \dot{\theta}_1^2 \sin\theta_1)(L_2 \cos\theta_2) + (m_2 g) L_2 \sin\theta_2 = 0$$

$$(m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\ddot{\theta}(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$
$$+ (m_2 L_1 L_2)\dot{\theta}_1^2(\cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2)$$
$$+ (m_2 g L_2)\sin\theta_2 = 0$$

Useful trig ID's:

$$\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$$
$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$$
$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

then:

$$(m_2 L_2^2)\ddot{\theta}_2 + (m_2 L_1 L_2)\ddot{\theta}\cos(\theta_2 - \theta_1)$$
$$+ (m_2 L_1 L_2)\dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + (m_2 g L_2)\sin\theta_2 = 0$$

Equation exactly matches that on pg 6

MASS 1:

Sum forces $\perp$ to T_1 from FBD of m_1

$$m_1(L_1\ddot{\theta}_1) + (m_1 g)\sin\theta_1 - T_2\sin(\theta_2-\theta_1) = 0$$

Need another eqn to cancel T_2 - use FBD of m_2

$$T_2 - m_2(L_2\dot{\theta}_2^2) - m_2(L_1\dot{\theta}_1^2)\cos(\theta_2-\theta_1)$$
$$+ m_2(L_1\ddot{\theta}_1)\sin(\theta_2-\theta_1) - m_2 g\cos\theta_2 = 0$$

$$T_2 = (m_2L_2)\dot{\theta}_2^2 + (m_2L_1)\dot{\theta}_1^2\cos(\theta_2-\theta_1)$$
$$- m_2(L_1\ddot{\theta}_1)\sin(\theta_2-\theta_1) + m_2 g\cos\theta_2 = 0$$

Combining

$$(m_1L_1)\ddot{\theta}_1 + (m_1 g)\sin\theta_1 - (m_2L_2)\dot{\theta}_2^2\sin(\theta_2-\theta_1)$$
$$- m_2L_1\dot{\theta}_1^2\cos(\theta_2-\theta_1)\sin(\theta_2-\theta_1) \qquad (a)$$
$$+ m_2(L_1\ddot{\theta}_1)\sin(\theta_2-\theta_1)\sin(\theta_2-\theta_1)$$
$$- m_2 g\cos\theta_2\sin(\theta_2-\theta_1) = 0$$

Also from m_2:

$$(m_2L_2)\ddot{\theta}_2 + (m_2L_1)\dot{\theta}_1^2\sin(\theta_2-\theta_1) + m_2L_1\ddot{\theta}_1\cos(\theta_2-\theta_1)$$
$$+ m_2 g\sin\theta_2 = 0$$

$$(m_2L_2)\ddot{\theta}_2\cos(\theta_2-\theta_1) + (m_2L_1)\dot{\theta}_1^2\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)$$
$$+ m_2L_1\ddot{\theta}_1\cos^2(\theta_2-\theta_1) + m_2 g\sin\theta_2\cos(\theta_2-\theta_1) = 0$$
$$(b)$$

Adding (a) & (b)

$$(m_1 L_1)\ddot{\theta}_1 + (m_2 L_2)\ddot{\theta}_2 \cos(\theta_2-\theta_1) + (m_2 L_1)\ddot{\theta}_1 \sin^2(\theta_2-\theta_1)$$
$$+ (m_2 L_1)\ddot{\theta}_1 \cos^2(\theta_2-\theta_1) - (m_2 L_2)\dot{\theta}_2^2 \sin(\theta_2-\theta_1)$$
$$\cancel{- (m_2 L_1)\dot{\theta}_1^2 \cos(\theta_2-\theta_1)\sin(\theta_2-\theta_1)}$$
TERMS CANCEL
$$\cancel{+ (m_2 L_1)\dot{\theta}_1^2 \sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)} + (m_1 g)\sin\theta_1$$
$$- m_2 g \cos\theta_2 \sin(\theta_2-\theta_1) + m_2 g \sin\theta_2 \cos(\theta_2-\theta_1) = 0$$

Trig ID: $\cos^2 a + \sin^2 a = 1$

$$(m_1 L_1)\ddot{\theta}_1 + (m_2 L_2)\ddot{\theta}_2 \cos(\theta_2-\theta_1) + (m_2 L_1)\ddot{\theta}_1 \qquad (c)$$
$$- (m_2 L_2)\dot{\theta}_2^2 \sin(\theta_2-\theta_1) + (m_1 g)\sin\theta_1$$
$$+ m_2 g \underbrace{\left[-\cos\theta_2 \sin(\theta_2-\theta_1) + \sin\theta_2 \cos(\theta_2-\theta_1)\right]}_{(*)} = 0$$

Focusing on term in [(*)]:

$$-\cos\theta_2 \sin(\theta_2-\theta_1) + \sin\theta_2 \cos(\theta_2-\theta_1)$$
$$= -\cos\theta_2(\sin\theta_2 \cos\theta_1 - \cos\theta_2 \sin\theta_1)$$
$$+ \sin\theta_2(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1)$$
$$= \cancel{-\cos\theta_2 \sin\theta_2 \cos\theta_1} + \cos^2\theta_2 \sin\theta_1$$
$$\cancel{+ \cos\theta_2 \sin\theta_2 \cos\theta_1} + \sin^2\theta_2 \sin\theta_1$$
TERMS CANCEL
$$= \sin\theta_1(\cos^2\theta_2 + \sin^2\theta_2)$$
$$= \sin\theta_1$$

So (c) becomes:

$$(m_1 L_1)\ddot{\theta}_1 + (m_2 L_2)\ddot{\theta}_2 \cos(\theta_2 - \theta_1) + (m_2 L_1)\ddot{\theta}_1$$
$$-(m_2 L_2)\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + (m_1 g)\sin\theta_1 + (m_2 g)\sin\theta_1 = 0$$

Multiply equation by L_1

$$\boxed{\begin{aligned}&(m_1 + m_2)L_1^2\ddot{\theta}_1 + (m_2 L_1 L_2)\ddot{\theta}_2 \cos(\theta_2 - \theta_1)\\ &\quad - (m_2 L_1 L_2)\dot{\theta}_2^2 \sin(\theta_2 - \theta_1) + (m_1 + m_2) g L_1 \sin\theta_1 = 0\end{aligned}}$$

Equation exactly matches the one on pg 5

<u>SUMMARIZING NON-LINEAR EQNS</u>