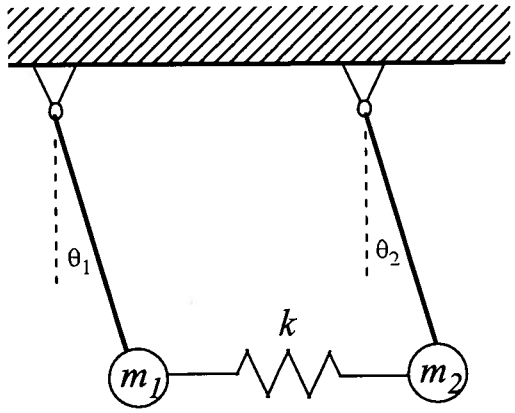
Problem 1: (Old Exam Problem - 25 points) You are given the linked pendulum system shown below, with degrees-of-freedom θ_1 and θ_2 measured counter-clockwise (However, your solution must include a full definition of these dofs!). The length of the bars supporting the masses are both L. Assume small angles throughout the problem.

(a) Determine the EoM for the following system using equilibrium (Newton or D'Alembert).

(b) Determine the EoM for the following system using LaGrange's equation.

(c) How do the two equations from parts (a) and (b) compare?



Assumption: small angles

VARIABLES

θ_1 = CCW rotation of bar supporting m_1 from vertical

θ_2 = " " " " m_2 " "

Δ = deformation of spring between m_1 & m_2

DOFs: θ_1 & θ_2 as defined above

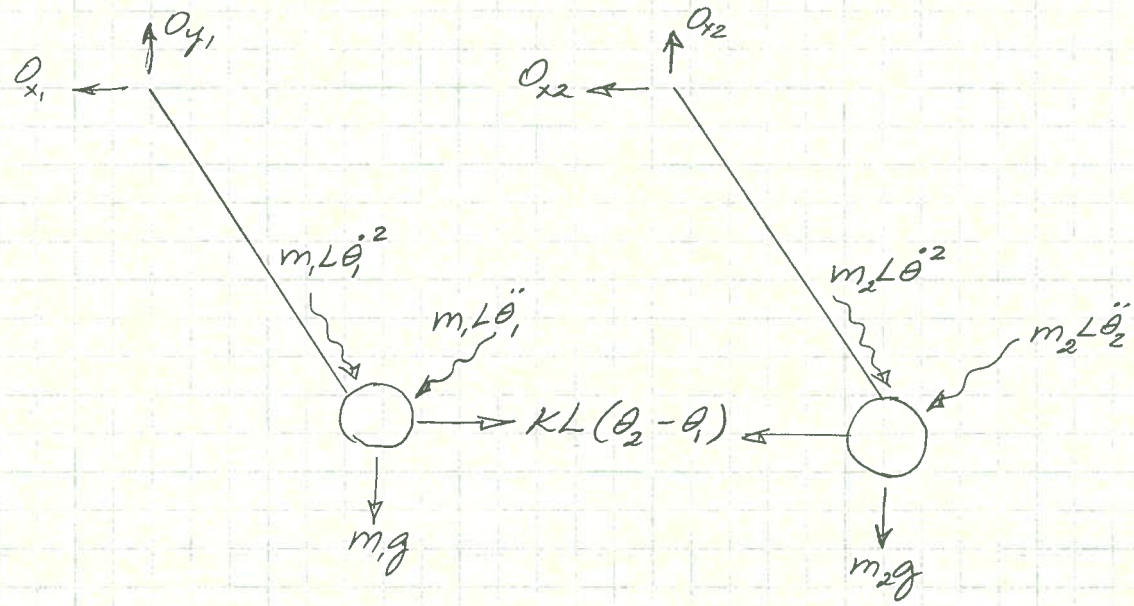
KINEMATICS

$\Delta = L\theta_2 - L\theta_1 = L(\theta_2 - \theta_1)$ assuming $L\theta_2 > L\theta_1$ ∴ spring in tension

(a) EoM using Equilibrium

- draw FBD using total forces since pendulums

FBD's - Total



Taking Moments About Supports

$$m_1L^2\ddot{\theta}_1 + m_1gL\sin\theta_1 - KL(\theta_2-\theta_1)L\cos\theta_1 = 0$$

$$m_2L^2\ddot{\theta}_2 + m_2gL\sin\theta_2 + KL(\theta_2-\theta_1)L\cos\theta_2 = 0$$

Assuming small angles: $\sin\theta \approx \theta$ $\cos\theta \approx 1$

$$(m_1L^2)\ddot{\theta}_1 + (KL^2+m_1gL)\theta_1 - (KL^2)\theta_2 = 0$$

$$(m_2L^2)\ddot{\theta}_2 + (KL^2+m_2gL)\theta_2 - (KL^2)\theta_1 = 0$$

Placing in matrix form:

$$\boxed{\begin{bmatrix} (m_1L^2) & 0 \\ 0 & (m_2L^2) \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (KL^2+m_1gL) & -KL^2 \\ -KL^2 & (KL^2+m_2gL) \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

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(b) EoM using LaGrange's

$$T = \frac{1}{2}m_1(L\dot{\theta}_1)^2 + \frac{1}{2}m_2(L\dot{\theta}_2)^2$$

$$V = \frac{1}{2}K[L(\theta_2-\theta_1)]^2 + m_1gL(1-\cos\theta_1) + m_2gL(1-\cos\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} - Q = 0$$

- for: $\underline{\underline{q_i = \theta_1}}$

$$\frac{\partial T}{\partial \theta_1} = 0$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = m_1(L\dot{\theta}_1)L = m_1L^2\dot{\theta}_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = \frac{d}{dt}\left(m_1L^2\dot{\theta}_1\right) = (m_1L^2)\ddot{\theta}_1$$

$$\frac{\partial V}{\partial \theta_1} = K[L(\theta_2-\theta_1)](-L) + m_1gL\underbrace{\sin\theta_1}_{\approx\theta_1}$$

$$= (KL^2 + m_1gL)\theta_1 - (KL^2)\theta_2$$

$$Q = 0$$

So: $(m_1L^2)\ddot{\theta}_1 + (KL^2 + m_1gL)\theta_1 - (KL^2)\theta_2 = 0$ (1)

- for: $\underline{\underline{q_i = \theta_2}}$

$$\frac{\partial T}{\partial \theta_2} = 0$$

$$\frac{\partial T}{\partial \dot{\theta}_2} = m_2L^2\dot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = (m_2L^2)\ddot{\theta}_2$$

$$\frac{\partial V}{\partial \theta_2} = K[L(\theta_2-\theta_1)](L) + m_2gL\underbrace{\sin\theta_2}_{\approx\theta_2} = (KL^2+m_2gL)\theta_2 - KL^2\theta_1$$

$Q = 0$

So: $(m_2L^2)\ddot{\theta}_2 - (KL^2)\theta_1 + (KL^2 + m_2gL)\theta_2 = 0$ ②

Placing ① & ② in matrix form:

$$\boxed{\begin{bmatrix} (m_1L^2) & 0 \\ 0 & (m_2L^2) \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} (KL^2 + m_1gL) & -KL^2 \\ -KL^2 & (KL^2 + m_2gL) \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}}$$

c) Equations found in parts (a) and (b) are exactly the same, as expected.